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ENGINEERING MATHEMATICS - I

for the

FIRST SEMESTER B.E. COURSE OF V.T.U

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PREFACE

I am very happy to present this student friendly text book **ENGINEERING MATHEMATICS-I** *on this auspicious day of Vinayaka Chathurthi. This self explanatory comprehensive book is in accordance with the VTU syllabus w.e.f the year 2010-11. It also caters to the need of autonomous institutions in Karnataka and other technological universities in the country. Student requirement and aspirations are the prime factors kept in mind in the compilation of this book in a lucid way.*

Two of my conscious senior colleagues in the department **Dr. D. Mamta** *and* **Ms. G. V. Pankaja** *have shouldered the responsibility in my effort to make this book error free. I am highly thankful to them in this regard.*

I am very much indebted to **Mr. K.V. Balakrishna** *of* **M/s. Sudha Publications** *for the immense confidence in my authorship, continuously publishing my books for the last 14 years accepting all my suggestions.*

Systematic approach by way of computer type setting work racing with the time by **Sri. S. Raghunandan** *and his team of* **M/s. Allkind** *is highly commendable.*

I thank the printers for the quality offset printing.

I am confident that the esteemed readers will bestow the same kind of response as in the past. I will humbly receive all the comments and valuable suggestions from the community of readers.

September 11th 2010 **K.S.Chandrashekar**

Mysore - 8

REWARD

VTU students of the current scheme w.e.f 2010-11, scoring 125/125 in all four papers of Engineering Mathematics I to IV Semesters (10 MAT 11, 21, 31, 41) will be rewarded with a cash prize of Rs.7,500/- by the author. Please write to the author directly along with attested xerox copies of marks cards of all the four semesters.

Achiever : *Mr. Bharath M.V,* **a student from 2006 batch of E&C branch from PESIT, Bangalore, received cash prize during 2008.**

SYLLABUS

ENGINEERING MATHEMATICS - I

Code	**: 10 MAT 11**	**IA Marks**	**: 25**
Hrs/Week	**: 04**	**Exam Hrs**	**: 03**
Total Hrs	**: 52**	**Exam Marks**	**: 100**

PART - A

Unit - I : Differential Calculus - 1

Determination of n^{th} derivative of standard functions (Illustrative examples). Leibnitz's theorem (without proof) and problems.

Rolle's theorem - Geometrical interpretation. Lagrange's and Cauchy's mean value theorems. Taylor's and Maclaurin's-series expansions of functions of one variable (without proof). **[6 hours]**

Unit - II : Differential Calculus - 2

Indeterminate forms - L'Hospital's rule (without proof), Polar curves : Angle between polar curves; Pedal equation for polar cuves. Derivative of arc length - concept and formulae without proof. Radius of curvature - Cartesian, parametric, polar and pedal forms. **[7 hours]**

Unit - III : Differential Calculus - 3

Partial differentiation : Partial derivatives, total derivative and chain rule.
Jacobians - direct evaluation.

Taylor's expansion of a function of two variables-Illustrative examples. Maxima and Minima for function of two variables, Applications - Errors and Approximations.

[6 hours]

Unit - IV : Vector Calculus

Scalar and vector point functions - Gradient, Divergence, Curl, Laplacian, Solenoidal and Irrotational vectors. Vector Identities : div $(\phi \vec{A})$, Curl $(\phi \vec{A})$, Curl $(\mathit{grad}\ \phi)$, div (Curl $\vec{A}$), div $(\vec{A} \times \vec{B})$ and Curl (Curl $\vec{A}$).

Orthogonal Curvilinear Coordinates - Definition, unit vectors, scale factors, orthogonality of cylindrical and spherical systems. Expression for Gradient, Divergence, Curl, Laplacian in an orthogonal system and also in cartesian, cylindrical and spherical Systems as particular cases - No problems. **[7 hours]**

PART - B

Unit-V : Integral Calculus

Differentiation under the integral sign-simple problems with constant limits. Reduction formulae for the integrals of $\sin^n x$, $\cos^n x$, $\sin^m x \cos^n x$ and evaluation of these
integrals with standard limits - problems.

Tracing of curves in cartesian, parametric and polar forms - Illustrative examples. Applications - Area, Perimeter, Surface area and Volume. Computation of these in respect of the curves - (i) Astroid : $x^{2/3} + y^{2/3} = a^{2/3}$ (ii) Cycloid : $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ (iii) Cardioide : $r = a(1 + \cos\theta)$

[6 hours]

Unit-VI : Differential Equations

Solution of first order and first degree equations : Recapitulation of the method of separation of variables with illustrative examples. Homogeneous, Exact, Linear equations and reducible to these forms, Applications : orthogonal trajectories.

[7 hours]

Unit-VII : Linear Algebra - 1

Recapitulation of Matrix Theory. Elementary transformations, Reduction of the given matrix to echelon and normal forms, Rank of a matrix, consistency of a system of linear equations and solution. Solution of a system of linear homogeneous equations (trivial and non-trivial solutions). Solution of a system of non- homogeneous equations by Gauss elimination and Gauss-Jordan methods. **[6 hours]**

Unit-VIII : Linear Algebra - 2

Linear transformations. Eigen values and eigen vectors of a square matrix, similarity of matrices, Reduction to diagonal form, Quadratic forms, Reduction of quadratic form into canonical form, Nature of quadratic forms. **[7 hours]**

Note : In the case of Illustrative Examples, questions are not to be set.

CONTENTS

PART - A

PART - A

Unit - I

DIFFERENTIAL CALCULUS - 1

1.1 Successive Differentiation

1.11 Introduction

As we all know, the meaning of the word 'successive' is one after another or again and again. This topic deals with differentiation of a given function of a single variable again and again.

1.12 Successive (Higher order) Derivatives

If $y = f(x)$ we know that $\dfrac{dy}{dx} = f'(x)$ is called as the derivative of y *w.r.t* x.

As we have the objective of differentiating again and again this shall be called as the first derivative of y *w.r.t* x. Symbolically we write it as

$$y_1 = \frac{dy}{dx} = Dy = f'(x), \text{ where } D = \frac{d}{dx}$$

The derivative of the first derivative is called as the second derivative of y *w.r.t* x.

ie., $$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = \{f'(x)\}'$$

or $$y_2 = \frac{d^2 y}{dx^2} = D^2 y = f''(x), \text{ where } D^2 = \frac{d^2}{dx^2}$$

So, in general the derivative of the $(n-1)^{th}$ derivative of y *w.r.t* x is called as the n^{th} **derivative of** y *w.r.t* x. Symbolically

$$y_n = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right) = D(D^{n-1} y) = \left\{f^{(n-1)}(x)\right\}'$$

Thus $y_n = \dfrac{d^n y}{dx^n} = D^n y = f^{(n)}(x)$ where $D^n = \dfrac{d^n}{dx^n}$, represents the n^{th} **derivative** of y *w.r.t* x. $y_1, y_2 \ldots$ are respectively called as the derivatives of order one, two, ... etc. $y = f(x)$ is equivalent to $y_0 = f^{(0)}(x)$ and this can be regarded as the derivative of order zero.

If an expression represents the n^{th} derivative of a function obviously it must give first derivative, second derivative .... corresponding to $n = 1, 2, \ldots$ Now the question is, can we find the n^{th} derivative of any function? The answer is *no*. The answer for this question is *yes* if only we are able to notice a *sequential change* from one derivative to the other. Suppose we have numbers $1, 4, 9, 16, \ldots$ and try to guess the n^{th} term, it is possible if only we re-write them as $1^2, 2^2, 3^2, 4^2 \ldots$ so that the n^{th} term of the sequence is n^2. Similarly the numbers $1, 2, 6, 24, 120, \ldots$ should be put in the form $1!, 2!, 3!, 4!, 5!, \ldots$ so that the n^{th} term of the sequence is $n!$. So we should have a sequence or create a sequence (*by alternative representation*) from one derivative to the succeeding derivative for deriving the n^{th} derivative of a given function which definitely gives first, second, third, ... derivatives of the given function corresponding to $n = 1, 2, 3 \ldots.$

We shall now proceed to derive the n^{th} derivatives of some standard functions.

1.13 n^{th} derivatives of some standard functions

$$y_1 = \frac{dy}{dx} = a\,e^{ax}; \quad y_2 = \frac{d}{dx}(a\,e^{ax}) = a \cdot a\,e^{ax} = a^2 e^{ax}$$

$$y_3 = \frac{d}{dx}(a^2 e^{ax}) = a^3 e^{ax}$$

(*We should notice the sequence $a = a^1$ in y_1, a^2 in y_2, a^3 in y_3 .... with the factor e^{ax} fixed*)

Thus $\boxed{y_n = D^n(e^{ax}) = a^n e^{ax}}$

We know that $\dfrac{d}{dx}(a^x) = a^x \log a$

$$y_1 = \frac{d}{dx}(a^{mx}) = a^{mx}\log a \cdot \frac{d}{dx}(mx) = m\log a \cdot a^{mx}$$

$$y_2 = m\log a \cdot \frac{d}{dx}(a^{mx}) = m\log a \cdot (m\log a \cdot a^{mx})$$

ie., $\quad y_2 = (m\log a)^2 a^{mx}$

$\quad y_3 = (m\log a)^3 a^{mx}$ and so on.

Thus $\boxed{y_n = D^n(a^{mx}) = (m\log a)^n a^{mx}}$

We know that $a = e^{\log a}$ $\quad \therefore \quad y = a^{mx} = (e^{\log a})^{mx} = (e^{m \log a})^x$

i.e., $\quad y = e^{bx}$ where $b = m \log a$

Hence $y_n = b^n e^{bx}$ [*by result* (1)]

Thus $\boxed{D^n (a^{mx}) = (m \log a)^n a^{mx}}$

where m is a positive integer and $m > n$

$$y_1 = m(ax+b)^{m-1} \cdot a$$

$$y_2 = m(m-1)(ax+b)^{m-2} \cdot a^2$$

$$y_3 = m(m-1)(m-2)(ax+b)^{m-3} a^3$$

(It may be noticed that the factors of m is being accumulated according to the order of the derivative. Further the ending factor is m $(m-0)$ *in respect of* y_1, $(m-1)$ *in respect of* y_2, $(m-2)$ *in respect of* y_3 *etc. Obviously it must be* $[m-(n-1)]$ *in respect of* y_n*)*

Thus $\boxed{y_n = m(m-1)(m-2)\ldots[m-(n-1)](ax+b)^{m-n} a^n}$

Let us write $y = (ax+b)^{-1}$

$$y_1 = -1(ax+b)^{-2} \cdot a = (-1)^1 1!(ax+b)^{-2} \cdot a$$

$$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2 = (-1)^2 2!(ax+b)^{-3} \cdot a^2$$

$$y_3 = (-1)(-2)(-3)(ax+b)^{-4} a^3 = (-1)^3 3!(ax+b)^{-4} a^3$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$y_n = (-1)^n n!(ax+b)^{-(n+1)} a^n$$

Thus $\boxed{y_n = D^n \left[\dfrac{1}{ax+b} \right] = \dfrac{(-1)^n n! a^n}{(ax+b)^{n+1}}}$

$$y_1 = \frac{1}{ax+b} \cdot a = a(ax+b)^{-1}$$

$$y_2 = a(-1)(ax+b)^{-2} \cdot a = a^2(-1)^1 1!(ax+b)^{-2}$$

$$y_3 = a^2(-1)(-2)(ax+b)^{-3} \cdot a = a^3(-1)^2 2!(ax+b)^{-3}$$

Similarly $y_4 = a^4(-1)^3 3!(ax+b)^{-4}$

(It must be noticed that the expression for y_1 can be written as $a^1(-1)^0 0!(ax+b)^{-1}$ since $(-1)^0 = 1 = 0!$. With this it is convenient to notice the sequence in R.H.S corresponding to $y_1, y_2, y_3, y_4, \dots$)

$$\therefore \quad y_n = a^n(-1)^{n-1}(n-1)!(ax+b)^{-n}$$

Thus $$\boxed{y_n = D^n[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}}$$

$$y_1 = \cos(ax+b) \cdot a$$

[The successive derivatives will be $-a^2\sin(ax+b)$, $-a^3\cos(ax+b)$, $a^4\sin(ax+b)$ and so on. It is not possible to create a sequence in this manner. Hence we need to put y_1 similar to y by using the trigonometric allied angle formula that $\cos\theta = \sin(\pi/2+\theta)$]

Thus $\quad y_1 = a\sin\left[\frac{\pi}{2} + ax + b\right]$

Now $\quad y_2 = a\cos\left[\frac{\pi}{2} + ax + b\right] \cdot a$

or $\quad y_2 = a^2\sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + ax + b\right)\right]$

$$y_2 = a^2\sin\left(2 \cdot \frac{\pi}{2} + ax + b\right).$$

Similarly $y_3 = a^3\sin\left(3 \cdot \frac{\pi}{2} + ax + b\right)$

Thus $$\boxed{y_n = D^n[\sin(ax+b)] = d^n\sin\left(n\frac{\pi}{2} + ax + b\right)}$$

$$y_1 = -a \sin(ax+b) \quad \text{But} \quad -\sin\theta = \cos(\pi/2+\theta)$$

$$y_1 = a\cos\left(\frac{\pi}{2}+ax+b\right)$$

$$y_2 = -a^2\sin\left(\frac{\pi}{2}+ax+b\right)$$

ie., $$y_2 = a^2\cos\left[\frac{\pi}{2}+\left(\frac{\pi}{2}+ax+b\right)\right]$$

$$y_2 = a^2\cos\left(2\cdot\frac{\pi}{2}+ax+b\right).$$

Similarly $$y_3 = a^3\cos\left(3\cdot\frac{\pi}{2}+ax+b\right)$$

Thus $$\boxed{y_n = D^n[\cos(ax+b)] = a^n\cos\left(n\frac{\pi}{2}+ax+b\right)}$$

$$y_1 = e^{ax}\cdot b\cos(bx+c)+a\,e^{ax}\sin(bx+c),$$

by applying product rule.

ie., $$y_1 = e^{ax}[\,b\cos(bx+c)+a\sin(bx+c)\,]$$

(The number of terms in the successive derivatives which is 2 in y_1 will become 4 in y_2, 8 in y_3 etc. if we proceed as it is. We have to plan to put y_1 similar to the form of y. This is possible by a special substitution)

Let us take the substitution $a = r\cos\theta$, $b = r\sin\theta$ as it is possible to express the newly introduced variables r and θ in terms of a and b by simple elimination.

Squaring and adding we get $a^2+b^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

Dividing we get $b/a = \tan\theta$

Hence $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

We shall use the substitution for the constants a and b present in the R.H.S of y_1 at only two places so that we can simplify. Thus

$$y_1 = e^{ax}[\,r\sin\theta\cos(bx+c)+r\cos\theta\sin(bx+c)\,]$$

ie., $y_1 = r\,e^{ax}\sin(\theta+bx+c)$, where we have used the formula

$$\sin A \cos B + \cos A \sin B = \sin(A + B)$$

(It can be seen that y_1 has assumed a form similar to that of y.)

Differentiating again and simplifying as before we can obtain

$$y_2 = r^2 e^{ax} \sin(2\theta + bx + c).$$

Similarly $\quad y_3 = r^3 e^{ax} \sin(3\theta + bx + c)$

Thus $\quad y_n = r^n e^{ax} \sin(n\theta + bx + c)$,

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

Thus $\boxed{D^n [e^{ax} \sin(bx + c)] = (\sqrt{a^2 + b^2})^n e^{ax} \sin[n \tan^{-1}(b/a) + bx + c]}$

$$y_1 = e^{ax} \cdot -b\sin(bx + c) + a e^{ax} \cos(bx + c), \text{ by product rule.}$$

ie., $\quad y_1 = e^{ax} [a\cos(bx + c) - b\sin(bx + c)]$

Let us put $a = r\cos\theta$, and $b = r\sin\theta$.

$\therefore \quad a^2 + b^2 = r^2$ and $\tan\theta = b/a$

ie., $\quad r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

Now, $\quad y_1 = e^{ax} [r\cos\theta\cos(bx + c) - r\sin\theta\sin(bx + c)]$

ie., $\quad y_1 = r e^{ax} \cos(\theta + bx + c)$ where we have used the formula
$\cos A \cos B - \sin A \sin B = \cos(A + B)$

Differentiating again and simplifying as before,

$$y_2 = r^2 e^{ax} \cos(2\theta + bx + c).$$

Similarly $y_3 = r^3 e^{ax} \cos(3\theta + bx + c)$.

Thus $\quad y_n = r^n e^{ax} \cos(n\theta + bx + c)$,

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

Thus $\boxed{D^n [e^{ax} \cos(bx + c)] = (\sqrt{a^2 + b^2})^n e^{ax} \cos[n \tan^{-1}(b/a) + bx + c]}$

5. $y = \log(2x+5) \quad \therefore \; y_n = \dfrac{(-1)^{n-1}(n-1)!\,2^n}{(2x+5)^n} \quad \ldots (a = 2,\; b = 5$ in F

6. $y = \dfrac{1}{(x+1)^2}$ or $y = (x+1)^{-2} \quad (m = -2,\; a = 1,\; b = 1$ in $F_3)$

$y_n = \{(-2)(-3)(-4)\ldots(-2-(n-1))\}(x+1)^{-2-n} \cdot 1^n$

ie., $\quad y_n = (-1)^n \cdot 2.\,3.\,4\ldots(n+1)(x+1)^{-(n+2)}$. Here $1^n = 1$

(Observe that -1 is a common factor in all the n terms)

$$y_n = \frac{(-1)^n (n+1)!}{(x+1)^{n+2}}$$

7. $y = \cos(4x+3) \quad \therefore \; y_n = 4^n \cos\left(\dfrac{n\pi}{2} + 4x + 3\right) \quad \ldots (a = 4,\; b = 3$ in $F_7)$

8. $y = \sin 6x \quad \therefore \; y_n = 6^n \sin\left(\dfrac{n\pi}{2} + 6x\right) \quad \ldots (a = 6,\; b = 0$ in $F_6)$

9. $y = e^{2x}\cos 3x \quad \ldots \quad (a = 2,\; b = 3,\; c = 0$ in $F_9)$

$y_n = (\sqrt{13})^n e^{2x} \cos[\,n \tan^{-1}(3/2) + 3x\,]$

10. $y = e^x \sin x \quad \ldots \quad (a = 1,\; b = 1,\; c = 0$ in $F_8)$

$y_n = (\sqrt{2})^n e^x \sin(n \tan^{-1} 1 + x) = (\sqrt{2})^n e^x \sin\left(\dfrac{n\pi}{4} + x\right)$

*** 11. $y = \log_{10}(4x^2 - 1)$

It is important to note that we have to convert the logarithm to the base e by the property

$$\boxed{\log_{10} x = \frac{\log_e x}{\log_e 10}}$$

$$y = \frac{\log_e(4x^2-1)}{\log_e 10} = \frac{\log_e(2x-1) + \log_e(2x+1)}{\log_e 10}$$

Thus $\quad y_n = \dfrac{1}{\log_e 10}\left\{\dfrac{(-1)^{n-1}(n-1)!\,2^n}{(2x-1)^n} + \dfrac{(-1)^{n-1}(n-1)!\,2^n}{(2x+1)^n}\right\}$

12. $y = \cos h^2 3x$

$$>> \quad y = \left[\frac{e^{3x} + e^{-3x}}{2} \right]^2 = \frac{1}{4} \left[e^{6x} + e^{-6x} + 2 \right]$$

$$\text{Thus} \quad y_n = \frac{1}{4} \left[6^n e^{6x} + (-6)^n \ e^{-6x} \right]$$

13. $y = e^{2x} \cos h \, 4x$

$$>> \quad y = e^{2x} \cdot \frac{1}{2} \left(e^{4x} + e^{-4x} \right) = \frac{1}{2} \left(e^{6x} + e^{-2x} \right)$$

$$\text{Thus} \quad y_n = \frac{1}{2} \left\{ 6^n e^{6x} + (-2)^n e^{-2x} \right\}$$

14. $y = \cos^2 x$

$$>> \quad y = \frac{1}{2} \left(1 + \cos 2x \right)$$

$$\text{Thus} \quad y_n = \frac{1}{2} \left\{ 0 + 2^n \cos \left(\frac{n\pi}{2} + 2x \right) \right\} = 2^{n-1} \cos \left(\frac{n\pi}{2} + 2x \right)$$

15. $y = \sin 8x \cdot \cos 5x$

$$>> \quad y = \frac{1}{2} \left[\sin (8x + 5x) + \sin (8x - 5x) \right] = \frac{1}{2} \left(\sin 13x + \sin 3x \right)$$

$$\text{Thus,} \quad y_n = \frac{1}{2} \left\{ (13)^n \sin \left(\frac{n\pi}{2} + 13x \right) + 3^n \sin \left(\frac{n\pi}{2} + 3x \right) \right\}$$

1.11 Leibnitz theorem for the n^{th} derivative of a product

Statement : *If u and v are functions of x then*

$$D^n (uv) = (uv)_n = uv_n + {}^nC_1 \, u_1 v_{n-1} + {}^nC_2 \, u_2 v_{n-2} + \ldots + u_n v$$

Working procedure for problems

- In problems, to find the n^{th} derivative of a given product involving polynomial functions like x, x^2, $(x+1)^3$ etc we have to take them as the first function 'u' since $D^n(u) = u_n$ will become zero for some n, with the result the process terminates at some stage.
- In some problems we have to use our discretion for getting the result in some specific form.

[illegible] PROBLEMS

1. $y = x^3\, 4^x$

Let $u = x^3,\ v = 4^x$

We have Leibnitz theorem,

$$(uv)_n = uv_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2\, u_2 v_{n-2} + \ldots + u_n v$$

But ${}^nC_1 = n,\ {}^nC_2 = \dfrac{n(n-1)}{1\cdot 2},\ {}^nC_3 = \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$

We also have $v_n = D^n(4^x) = (\log 4)^n\, 4^x$ and $u_1 = D(x^3) = 3x^2$

$$u_2 = D(3x^2) = 6x\ ;\ u_3 = D(6x) = 6,\ u_4 = D(6) = 0$$

Now applying Leibnitz theorem we have,

$$y_n = x^3\cdot(\log 4)^n\, 4^x + n\cdot 3x^2\cdot(\log 4)^{n-1}\, 4^x$$

$$+ \frac{n(n-1)}{2}\, 6x\,(\log 4)^{n-2}\, 4^x + \frac{n(n-1)(n-2)}{6}\, 6\cdot(\log 4)^{n-3}\cdot$$

Thus $y_n = (\log 4)^{n-3}\cdot 4^x\ \ x^3(\log 4)^3 + 3nx^2(\log 4)^2$

$$+ 3n(n-1)\,x\,(\log 4) + n(n-1)(n-2)$$

2. $y = x^2\cos^2 3x$

$$= x^2\left(\frac{1+\cos 6x}{2}\right) = \frac{x^2}{2} + \frac{1}{2}\,(x^2\cos 6x)$$

$$y_n = D^n\left(\frac{x^2}{2}\right) + \frac{1}{2}\, D^n\,(x^2\cos 6x)$$

ie., $y_n = 0 + \dfrac{1}{2}\, D^n(x^2\cos 6x)$ where $n > 2$

Now, let $u = x^2$ and $v = \cos 6x$.

Applying Leibnitz theorem we get

$$y_n = \frac{1}{2}\left\{x^2 (\cos 6x)_n + n \cdot 2x \cdot (\cos 6x)_{n-1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 (\cos 6x)_{n-2}\right\}$$

We have $(\cos 6x)_n = D^n (\cos 6x) = 6^n \cos\left(\frac{n\pi}{2} + 6x\right)$

$$y_n = \frac{1}{2}\left\{x^2 \cdot 6^n \cos\left(\frac{n\pi}{2} + 6x\right) + 2nx \cdot 6^{n-1} \cos\left((n-1)\frac{\pi}{2} + 6x\right)\right.$$
$$\left. + n(n-1) \cdot 6^{n-2} \cos\left((n-2)\frac{\pi}{2} + 6x\right)\right\}$$

Thus $$y_n = \frac{6^{n-2}}{2}\left\{36x^2 \cos\left(\frac{n\pi}{2} + 6x\right)\right.$$
$$\left. + 12nx \cos\left((n-1)\frac{\pi}{2} + 6x\right) + n(n-1)\cos\left((n-2)\frac{\pi}{2} + 6x\right)\right\}$$

3. $y = x^2 \log 4x$

Let $u = x^2,\ v = \log 4x$

Applying Leibnitz theorem we have,

$$y_n = x^2 (\log 4x)_n + n \cdot 2x \cdot (\log 4x)_{n-1} + \frac{n(n-1)}{2} \cdot 2 (\log 4x)_{n-2}$$

We have $(\log 4x)_n = D^n (\log 4x)$

$$= \frac{(-1)^{n-1}(n-1)!\,4^n}{(4x)^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Now $$y_n = x^2 \cdot \frac{(-1)^{n-1}(n-1)!}{x^n} + 2nx \cdot \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$
$$+ n(n-1)\cdot\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}$$

Thus $$y_n = \frac{1}{x^{n-2}}\left\{(-1)^{n-1}(n-1)! + 2n(-1)^{n-2}(n-2)!\right.$$
$$\left. + n(n-1)(-1)^{n-3}(n-3)!\right\} \text{ where } n > 2$$

Since $(n-3)!$ is valid only when $n > 2$, we claim that the above expression for y_n is valid for $n > 2$

4. $y = x\, e^{2x} \sin 3x \cos 2x$

$$y = x\, e^{2x} \cdot \frac{1}{2} \left[\sin 5x + \sin x \right]$$

$$\therefore \quad y_n = \frac{1}{2} \left[D^n \left(x \cdot e^{2x} \sin 5x \right) + D^n \left(x \cdot e^{2x} \sin x \right) \right] \qquad \ldots (1)$$

We have to apply Leibnitz theorem separately for the two terms by taking $u = x$, $v = e^{2x} \sin 5x$ in the first term and $u = x$, $v = e^{2x} \sin x$ in the second term. Further we have,

$$v_n = (\sqrt{29}\,)^n e^{2x} \sin \left(n \tan^{-1} (5/2) + 5x \right) \text{ in the first term and}$$

$$v_n = (\sqrt{5}\,)^n e^{2x} \sin \left(n \tan^{-1} (1/2) + x \right) \text{ in the second term.}$$

Thus $$y_n = \frac{1}{2} \Big[x \cdot (\sqrt{29}\,)^n e^{2x} \sin \left\{ n \tan^{-1} (5/2) + 5x \right\}$$
$$+ n \cdot 1 \cdot (\sqrt{29}\,)^{n-1} e^{2x} \sin \left\{ (n-1) \tan^{-1} (5/2) + 5x \right\} \Big]$$
$$+ \frac{1}{2} \Big[x \cdot (\sqrt{5}\,)^n e^{2x} \sin \left\{ n \tan^{-1} (1/2) + x \right\}$$
$$+ n \cdot 1 \cdot (\sqrt{5}\,)^{n-1} e^{2x} \sin \left\{ (n-1) \tan^{-1} (1/2) + x \right\} \Big]$$

5. $y = e^{-x} x^2 \sinh 2x$

$$y = e^{-x} \cdot x^2 \cdot \frac{1}{2} \left(e^{2x} - e^{-2x} \right)$$

i.e., $$y = \frac{1}{2} \left(x^2 e^x - x^2 e^{-3x} \right)$$

$$\therefore \quad y_n = \frac{1}{2} \left[D^n \left(x^2 e^x \right) - D^n \left(x^2 e^{-3x} \right) \right]$$

Applying Leibnitz theorem to each of the two terms we get,

$$y_n = \frac{1}{2} \left[x^2 e^x + n \cdot 2x \cdot e^x + \frac{n(n-1)}{2} \cdot 2 \cdot e^x \right]$$
$$- \frac{1}{2} \left[x^2 (-3)^n e^{-3x} + n \cdot 2x \cdot (-3)^{n-1} e^{-3x} + \frac{n(n-1)}{2} \cdot 2 (-3)^{n-2} \right.$$

i.e., $$y_n = \frac{e^x}{2} \left[x^2 + 2nx + n(n-1) \right] - (-3)^{n-2} \frac{e^{-3x}}{2} \left[9x^2 - 6nx + n(n-1) \right]$$

6. $\quad y = e^x \log x$

Here neither of the functions is a polynomial and either of them could be the first function

Let us take $u = e^x$ and $v = \log x$

Applying Leibnitz theorem we have,

$$y_n = e^x D^n(\log x) + n\, e^x D^{n-1}(\log x)$$

$$+ \frac{n(n-1)}{2!} e^x D^{n-2}(\log x) + \cdots + D^n(e^x) \cdot \log x$$

$$y_n = e^x \cdot \frac{(-1)^{n-1}(n-1)!}{x^n} + n\, e^x \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$

$$+ \frac{n(n-1)}{2} e^x \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} + \cdots + e^x \log x$$

$$y_n = e^x \left\{ \frac{(-1)^{n-1}(n-1)!}{x^n} + \frac{n(-1)^{n-2}(n-2)!}{x^{n-1}} \right.$$

$$\left. + \frac{n(n-1)}{2} \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} + \cdots + \log x \right\}$$

>> Let $y = \dfrac{\log x}{x} = \log x \cdot \dfrac{1}{x}$ and let $u = \log x,\ v = \dfrac{1}{x}$

We have Leibnitz theorem,

$$(uv)_n = uv_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2\, u_2 v_{n-2} + \cdots + u_n v \qquad \ldots (1)$$

Now, $\quad u = \log x \quad \therefore \quad u_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

$\quad v = \dfrac{1}{x} \quad \therefore \quad v_n = \dfrac{(-1)^n\, n!}{x^{n+1}}$

Using these in (1) by taking appropriate values for n we get,

$$D^n\left(\frac{\log x}{x}\right) = \log x \cdot \frac{(-1)^n n!}{x^{n+1}} + n\,\frac{1}{x}\cdot\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$+\frac{n(n-1)}{1\cdot 2}\left(-\frac{1}{x^2}\right)\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}+\cdots+\frac{(-1)^{n-1}(n-1)!}{x^n}\cdot\frac{1}{x}$$

ie., $$= \log x \cdot \frac{(-1)^n n!}{x^{n+1}} + \frac{(-1)^{n-1} n!}{x^{n+1}} - \frac{(-1)^{n-2} n!}{2x^{n+1}} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x + (-1)^{-1} - \frac{(-1)^{-2}}{2} + \cdots + \frac{(-1)^{-1}(n-1)!}{n!}\right]$$

Note : $(-1)^{-1} = \dfrac{1}{-1} = -1$; $(-1)^{-2} = \dfrac{1}{(-1)^2} = 1$

Also $\dfrac{(n-1)!}{n!} = \dfrac{(n-1)!}{n\cdot(n-1)!} = \dfrac{1}{n}$

Thus $$\frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3}\cdots - \frac{1}{n}\right]$$

>> Let $y = x^n \log x$.

It is important to take a note that the given function is involved with n and hence the n^{th} derivative $y_n = D^n(x^n \log x)$ is to be viewed as follows.

$y_1 = D^1(x\log x)$; $y_2 = D^2(x^2\log x)$; $y_3 = D^3(x^3 \log x)$ etc.,

$n = 1, 2, 3\ldots$ is to be correlated both in L.H.S and R.H.S

Since we have to prove a result for positive integral values of n the method of mathematical induction is well suited

Step-1 : We shall verify the result when $n = 1$

$$\text{L.H.S} = D^1(x\log x) = \frac{d}{dx}(x\log x) = x\cdot\frac{1}{x} + \log x = 1 + \log x$$

$$\text{R.H.S} = 1!\,(\log x + 1) = \log x + 1$$

L.H.S = R.H.S $\Rightarrow$ The result is true when $n = 1$

Step-2 We shall assume the result to be true for $n = k$, where k is any positive integer

$$D^k\left(x^k \log x\right) = k!\left\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right\} \quad \ldots (1)$$

Step-3 : We have to prove the result for $n = k+1$.

$$D^{k+1}\left(x^{k+1} \log x\right) = D^k\left[D\left(x^{k+1} \log x\right)\right]$$

$$= D^k\left[x^{k+1} \cdot \frac{1}{x} + (k+1)\, x^k \cdot \log x\right]$$

ie., $$D^{k+1}\left(x^{k+1} \log x\right) = D^k\left[x^k\right] + (k+1)\, D^k\left[x^k \log x\right] \quad \ldots (2)$$

Let us consider the first term $D^k\left[x^k\right]$

$$D(x^k) = kx^{k-1},\; D^2(x^k) = k(k-1)\,x^{k-2},$$

$$D^3(x^k) = k(k-1)(k-2)\,x^{k-3} \text{ etc.}$$

$$D^k(x^k) = k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1\, x^{k-k} = k! \cdot 1 = k!$$

$$D^k(x^k) = k!$$

Note : Remember $D^n(x^n) = n!$

Using this result and (1) in the R.H.S of (2) we have

$$D^{k+1}\left(x^{k+1} \log x\right) = k! + (k+1)\left\{k!\left(\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\right\}$$

ie., $$= k! + (k+1)!\left\{\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right\}$$

$$= (k+1)!\left\{\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{k!}{(k+1)!}\right\}$$

$$D^{k+1}\left(x^{k+1} \log x\right) = (k+1)!\left\{\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1}\right\} \quad \ldots (3)$$

Comparing (1) and (3) we conclude that the result is true for $n = k+1$.
Hence by the principle of mathematical induction the result is true for any positive integer n.

Thus $$D^n\left[x^n \log x\right] = n!\left\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right\}$$

Remark *The [illegible] arrive at the result as directed in the problem.*

>> $$y_n = D^n(x^n \log x) = D^{n-1}\{D(x^n \log x)\}$$

$$= D^{n-1}\left\{x^n \cdot \frac{1}{x} + n\,x^{n-1}\log x\right\}$$

$$= D^{n-1}\left(x^{n-1}\right) + n\,D^{n-1}\left(x^{n-1}\log x\right)$$

$y_n = (n-1)! + n\,y_{n-1}$. This proves the first part.

Now Putting the values for $n = 1, 2, 3$...we get

$$y_1 = 0! + 1 \cdot y_0 = 1 + \log x = 1!(\log x + 1)$$

$$y_2 = 1! + 2y_1 = 1 + 2(1 + \log x)$$

ie., $$y_2 = 2\log x + 3 = 2\left(\log x + 3/2\right) = 2!\left(\log x + 1 + \frac{1}{2}\right)$$

$$y_3 = 2! + 3y_2 = 2 + 3(2\log x + 3)$$

ie., $$y_3 = 6\log x + 11 = 6\left(\log x + 11/6\right) = 3!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right)$$

.. . . .

Thus $$y_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

>> Observing that the desired result is related to y_{n+1} being the $(n+1)^{th}$ derivative, it will be convenient to first obtain y_1 and then differentiate n times.

Consider $y = x^n \log x$ $\quad\therefore\quad$ $y_1 = x^n \cdot \dfrac{1}{x} + n\,x^{n-1}\log x = x^{n-1} + n\,x^{n-1}\log x$

Multiplying by x we get,

$$x\,y_1 = x^n + n\,x^n \log x = x^n + ny$$

Now, $D^n(x\,y_1) = D^n(x^n) + n\,D^n(y)$

Applying Leibnitz theorem to L.H.S and using $D^n(x^n) = n!$ we have,

$$x\,D^n(y_1) + n_{C_1} \cdot 1 \cdot D^{n-1}(y_1) = n! + n\,y_n$$

ie., $$x\,y_{n+1} + n\,y_n = n! + n\,y_n$$

$\Rightarrow$ $$x\,y_{n+1} = n! \quad \text{or} \quad y_{n+1} = \frac{n!}{x}$$

Note *The function involved in Examples 8, 9, 10 is the same and different technique is adopted in each problem to get the result as desired.*

11.

>> $$y = \tan x = \frac{\sin x}{\cos x}$$

$\cos x \cdot y = \sin x$ and differentiating n times we have,

$$D^n(\cos x \cdot y) = D^n(\sin x)$$

Applying Leibnitz theorem onto the L.H.S and making use of the standard formula in R.H.S we get,

$$\cos x \cdot y_n + {}^nC_1(-\sin x) \cdot y_{n-1} + {}^nC_2(-\cos x) \cdot y_{n-2}$$
$$+ {}^nC_3(\sin x) \cdot y_{n-3} + {}^nC_4(\cos x) \cdot y_{n-4} + \cdots = \sin(n\pi/2 + x) \quad \ldots (1)$$

Here $y_n, y_{n-1}, \ldots$ are all functions of x : $y_n(x), y_{n-1}(x)$ and $y_n(0)$ is to be understood as the value of y_n when $x = 0$. In this context, observing the desired result we need to put $x = 0$ in (1). $\sin 0 = 0, \cos 0 = 1$

Thus (1) now becomes,

$$y_n(0) - {}^nC_2\, y_{n-2}(0) + {}^nC_4\, y_{n-4}(0) - \cdots = \sin(n\pi/2)$$

Working procedure for

- Given y as a function of x (*explicit or implicit*) we need to establish a relation involving y_{n+2}, y_{n+1} and y_n or y_{n+1}, y_n and y_{n-1} etc.
- In order to establish such a relation we have to first establish a relation involving y_2, y_1, y or y_1, y as the case may be by differentiating and simplifying judiciously
- Later we have to differentiate the relation so obtained n times and Leibnitz theorem has to be employed in differentiating a product involved in the relation

Observe the following :

$$D^n(y_2) = y_{n+2}, \quad D^{n-1}(y_2) = y_{n+1}, \quad D^{n-2}(y_2) = y_n,$$
$$D^{n+1}(y_1) = y_{n+2}, \quad D^{n+1}(y) = y_{n+1} \text{ etc}$$

WORKED PROBLEMS

>> $$y = a\cos(\log x) + b\sin(\log x)$$

Part I

Differentiate *w.r.t* x

$$\therefore \quad y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cdot\cos(\log x)\cdot\frac{1}{x}$$ (we avoid quotient rule to find y_2

$$\Rightarrow \quad x\,y_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again *w.r.t* x we have,

$$x\,y_2 + 1\cdot y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

or $$x^2 y_2 + x\,y_1 = -\left[a\cos(\log x) + b\sin(\log x)\right] = -y$$

Thus $$\mathbf{x^2 y_2 + x\,y_1 + y = 0}$$

Now we have to differentiate this result n times

Part II

ie., $$D^n(x^2 y_2) + D^n(x\,y_1) + D^n(y) = 0$$

We have to employ Leibnitz theoreom for the first two terms
Hence we have,

$$\left\{x^2\cdot D^n(y_2) + n\cdot 2x\cdot D^{n-1}(y_2) + \frac{n(n-1)}{1\cdot 2}\cdot 2\cdot D^{n-2}(y_2)\right\}$$
$$+\left\{x\cdot D^n(y_1) + n\cdot 1\cdot D^{n-1}(y_1)\right\} + y_n = 0$$

ie., $$\left\{x^2 y_{n+2} + 2n\,x\,y_{n+1} + n(n-1)\,y_n\right\} + \left\{x\,y_{n+1} + ny_n\right\} + y_n = 0$$

ie., $$x^2 y_{n+2} + 2n\,x\,y_{n+1} + n^2 y_n - ny_n + x\,y_{n+1} + ny_n + y_n = 0$$

Thus $$\mathbf{x^2 y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2+1)\,y_n = 0}$$

$$\tan y = x \Rightarrow y = \tan^{-1} x$$

$$\therefore \quad y_1 = \frac{1}{1+x^2} \quad \text{or} \quad (1+x^2)\, y_1 = 1$$

Differentiating again *w.r.t* x we have

$$\mathbf{(1+x^2)\, y_2 + 2\, x\, y_1 = 0}$$

Now, differentiating this result n times we have,

$$D^n \left[(1+x^2)\, y_2 \right] + 2\, D^n\, [\, x\, y_1\,] = 0$$

Applying Leibnitz theorem to each of the terms we have,

$$(1+x^2)\, D^n (y_2) + n \cdot 2x \cdot D^{n-1} (y_2) + \frac{n(n-1)}{1 \cdot 2}\, 2\, D^{n-2} (y_2)$$

$$+ 2 \left\{ x \cdot D^n (y_1) + n \cdot 1 \cdot D^{n-1} (y_1) \right\} = 0$$

ie., $\quad \left\{ (1+x^2)\, y_{n+2} + n \cdot 2x \cdot y_{n+1} + (n^2 - n)\, y_n \right\} + 2 \left\{ x\, y_{n+1} + n y_n \right\} = 0$

ie., $\quad (1+x^2)\, y_{n+2} + 2(n+1)\, x\, y_{n+1} + (n^2 + n)\, y_n = 0$

Thus $\quad \mathbf{(1+x^2)\, y_{n+2} + 2\,(n+1)\, x\, y_{n+1} + n\,(n+1)\, y_n = 0}$

Note *The result of this example is related to the function* $y = \tan^{-1} x$ *which can be given in different versions as follows.*

(i) *If* $y = \tan^{-1} \left(\dfrac{1+x}{1-x} \right)$ *then show that*

$$(1+x^2)\, y_{n+2} + 2\,(n+1)\, x\, y_{n+1} + n\,(n+1)\, y_n = 0$$

>> Put $\quad x = \tan \theta$. Then $y = \tan^{-1} \left(\dfrac{1 + \tan \theta}{1 - \tan \theta} \right)$

ie., $\quad y = \tan^{-1} \tan (\pi/4 + \theta) = (\pi/4) + \theta \quad \text{or} \quad y = (\pi/4) + \tan^{-1} x$

This gives $y_1 = 0 + \dfrac{1}{1+x^2} \quad \text{or} \quad (1+x^2)\, y_1 = 1$

Then the result follows as in Problem - 13

(ii) *If* $y = \tan^{-1} \left(\dfrac{a+x}{a-x} \right)$*, then show that*

$$(a^2 + x^2)\, y_{n+2} + 2\,(n+1)\, x\, y_{n+1} + n\,(n+1)\, y_n = 0$$

>> $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$.

Put $x = a\tan\theta$

$\therefore \quad y = \tan^{-1}\left[\dfrac{a(1+\tan\theta)}{a(1-\tan\theta)}\right] = \tan^{-1}\tan(\pi/4+\theta) = (\pi/4)+\theta$

Thus $y = (\pi/4)+\theta$ where $x/a = \tan\theta$ or $\theta = \tan^{-1}(x/a)$

Now $y = (\pi/4)+\tan^{-1}(x/a)$

$\therefore \quad y_1 = 0 + \dfrac{1}{1+\dfrac{x^2}{a^2}}\cdot\dfrac{1}{a} = \dfrac{a^2}{a^2+x^2}\cdot\dfrac{1}{a} = \dfrac{a}{a^2+x^2}$ or $(a^2+x^2)\,y_1 = a$.

Differentiating *w.r.t* x we get,

$$(a^2+x^2)\,y_2 + 2\,x\,y_1 = 0 \qquad \ldots(1)$$

This part of the result is similar to the result in the first part of Problem - 13 as we have a^2 instead of 1. Proceeding on the same lines in applying Leibnitz theorem to both the terms of (1) we can can obtain,

$$(a^2+x^2)\,y_{n+2} + 2(n+1)\,x\,y_{n+1} + n(n+1)\,y_n = 0$$

14.

>> By data, $\cos^{-1}(y/b) = n\log(x/n)$ $\because$ $\log(a^m) = m\log a$

$\Rightarrow \quad \dfrac{y}{b} = \cos\left[n\log(x/n)\right]$ or $y = b\cdot\cos\left[n\log(x/n)\right] \qquad \ldots(1)$

Differentiating *w.r.t* x we get,

$$y_1 = -b\sin\left[n\log(x/n)\right]\cdot n\cdot\frac{1}{(x/n)}\cdot\frac{1}{n} \quad x\,y_1 = -n\,b\sin\left[n\log(x/n)\right]$$

Differentiating *w.r.t* x again we get,

$$x\,y_2 + 1\cdot y_1 = -n\cdot b\cos\left[n\log(x/n)\right]\cdot n\cdot\frac{1}{(x/n)}\cdot\frac{1}{n}$$

or $\quad x(x\,y_2+y_1) = -n^2 b\cos\left[n\log(x/n)\right] = -n^2 y$, by using (1).

or $\quad x^2 y_2 + x\,y_1 + n^2 y = 0$

Differentiating each term n times we have,

$$D^n(x^2 y_2) + D^n(x y_1) + n^2 D^n(y) = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{x^2 y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n\right\} + \left\{x\, y_{n+1} + n \cdot 1 \cdot y_n\right\} + n^2 y_n = 0$$

ie, $\quad x^2 y_{n+2} + 2nx\, y_{n+1} + n^2 y_n - n y_n + x\, y_{n+1} + n y_n + n^2 y_n = 0$

Thus $\quad \mathbf{x^2 y_{n+2} + (2n+1)\, x\, y_{n+1} + 2n^2 y_n = 0}$

15.

>> By data $y = \dfrac{\sinh^{-1} x}{\sqrt{1+x^2}}$

or $\quad \sqrt{1+x^2}\; y = \sinh^{-1} x$ *(we always avoid denominator for convenient differentiation)*

Now, differentiating *w.r.t* x we get,

$$\sqrt{1+x^2} \cdot y_1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \cdot y = \frac{1}{\sqrt{1+x^2}} \qquad \text{or} \qquad (1+x^2)\, y_1 + x y = 1$$

Differentiating *w.r.t* x again we get,

$$\left\{(1+x^2)\, y_2 + 2x\, y_1\right\} + \left\{x\, y_1 + 1 \cdot y\right\} = 0$$

ie., $\quad (1+x^2)\, y_2 + 3x\, y_1 + y = 0$

Differentiating each term n times we have

$$D^n\left[(1+x^2)\, y_2\right] + 3D^n\left[x\, y_1\right] + D^n(y) = 0$$

Applying Leibnitz theorem to the product terms we get

$$\left\{(1+x^2)\, y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n\right\} + 3\left\{x\, y_{n+1} + n \cdot 1 \cdot y_n\right\} + y_n = 0$$

ie, $\quad (1+x^2)\, y_{n+2} + 2nx\, y_{n+1} + n^2 y_n - n y_n + 3x\, y_{n+1} + 3n\, y_n + y_n = 0$

ie., $\quad (1+x^2)\, y_{n+2} + (2n+3)\, x\, y_{n+1} + (n^2 + 2n + 1)\, y_n = 0$

Thus $\quad \mathbf{(1+x^2)\, y_{n+2} + (2n+3)\, x\, y_{n+1} + (n+1)^2 y_n = 0}$

>> By data $y = \sin \log (x^2+2x+1)$

$\therefore$ $$y_1 = \cos \log (x^2+2x+1) \cdot \frac{1}{x^2+2x+1} \cdot 2x+2$$

ie., $$y_1 = \cos \log (x^2+2x+1) \cdot \frac{1}{(x+1)^2}\, 2(x+1)$$

ie., $$y_1 = \frac{2 \cos \log (x^2+2x+1)}{(x+1)} \quad \text{or} \quad (x+1)\, y_1 = 2 \cos \log (x^2+2x+1)$$

Differentiating *w.r.t* x again we get

$$(x+1)\, y_2 + 1 \cdot y_1 = -2 \sin \log (x^2+2x+1)\, \frac{1}{(x+1)^2} \cdot 2(x+1)$$

or $$(x+1)^2 y_2 + (x+1)\, y_1 = -4y \quad \text{or} \quad (x+1)^2 y_2 + (x+1)\, y_1 + 4y = 0$$

Differentiating each term n times we have,

$$D^n\left[(x+1)^2 y_2\right] + D^n\left[(x+1)\, y_1\right] + 4 D^n [y] = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{(x+1)^2 y_{n+2} + n \cdot 2(x+1) \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n\right\}$$
$$+ \left\{(x+1)\, y_{n+1} + n \cdot 1 \cdot y_n\right\} + 4y_n = 0$$

ie., $$(x+1)^2 y_{n+2} + 2n(x+1)\, y_{n+1} + n^2 y_n - n y_n + (x+1)\, y_{n+1} + n y_n + 4y_n = 0$$

Thus $$(x+1)^2 y_{n+2} + (2n+1)(x+1)\, y_{n+1} + (n^2+4)\, y_n = 0$$

>> $y = (x^2-1)^n$

Taking logarithms on both sides, . . . *(for convenience)*

$$\log y = n \log (x^2-1)$$

Differentiating w.r.t x we get

$$\frac{1}{y} y_1 = n \cdot \frac{1}{x^2 - 1} \cdot 2x \quad \text{or} \quad (x^2 - 1) y_1 = 2nxy$$

Differentiating again w.r.t x we get,

$$(x^2 - 1) y_2 + 2 x y_1 = 2n (x y_1 + y)$$

Now differentiating each term n times we have,

$$D^n \left[(x^2 - 1) y_2 \right] + 2D^n [x y_1] = 2n D^n [x y_1] + 2n D^n [y]$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{ (x^2 - 1) y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\}$$

$$+ 2 \{ x y_{n+1} + n \cdot 1 \cdot y_n \} = 2n \{ x y_{n+1} + n \cdot 1 \cdot y_n \} + 2n y_n$$

ie., $\quad (x^2 - 1) y_{n+2} + 2nx y_{n+1} + n^2 y_n - n y_n + 2 x y_{n+1} + 2n y_n$

$$= 2nx y_{n+1} + 2n^2 y_n + 2n y_n$$

ie., $\quad (x^2 - 1) y_{n+2} + 2x y_{n+1} - n^2 y_n - n y_n = 0$

ie., $\quad (x^2 - 1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0$

or $\quad \mathbf{(1 - x^2) y_{n+2} - 2x y_{n+1} + n(n+1) y_n = 0}$

Note : ***Alternative version of the problem :***

If $\quad y = (x^2 - 1)^n$ *show that* $y_n = \dfrac{d^n y}{dx^n}$ *satisfies the equation*

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1) y = 0$$

\>> We need to show that,

$$(1 - x^2) \frac{d^2}{dx^2} (y_n) - 2x \frac{d}{dx} (y_n) + n(n+1) y_n = 0$$

That is to show that, $(1 - x^2) y_n'' - 2x y_n' + n(n+1) y_n = 0$
and this is equivalent to

$$(1 - x^2) y_{n+2} - 2x y_{n+1} + n(n+1) y_n = 0$$

This is the same result established in Problem - 17

>> By data, $y = \log(x + \sqrt{1+x^2})$

$$y_1 = \frac{1}{(x+\sqrt{1+x^2})} \left\{ 1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \right\}$$

ie., $$y_1 = \frac{1}{(x+\sqrt{1+x^2})} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} \quad \text{or} \quad \sqrt{1+x^2}\, y_1 = 1$$

Differentiating *w.r.t* x again we get

$$\sqrt{1+x^2}\, y_2 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \cdot y_1 = 0 \quad \text{or} \quad (1+x^2)\, y_2 + x\, y_1 = 0$$

Now, $D^n\left[(1+x^2)\, y_2 \right] + D^n\left[x\, y_1 \right] = 0$

Applying Leibnitz theorem to each term we get,

$$\left\{ (1+x^2)\, y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\} + \left[x \cdot y_{n+1} + n \cdot 1 \cdot y_n \right] = 0$$

ie., $$(1+x^2)\, y_{n+2} + 2nx\, y_{n+1} + n^2 y_n - n y_n + x\, y_{n+1} + n y_n = 0$$

Thus $$\mathbf{(1+x^2)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + n^2 y_n = 0}$$

Note *Alternative version of the problem* The result of this problem also holds good if $y = \sinh^{-1} x$ since $\log(x+\sqrt{1+x^2}) = \sinh^{-1} x$

>> By data, $y = e^{m \cos^{-1} x}$

$$y_1 = e^{m\cos^{-1}x} \cdot \frac{-m}{\sqrt{1-x^2}} = \frac{-my}{\sqrt{1-x^2}} \quad \text{or} \quad \sqrt{1-x^2}\, y_1 = -my$$

Differentiating again we get,

$$\sqrt{1-x^2}\, y_2 + \frac{1}{2\sqrt{1-x^2}} (-2x)\, y_1 = -m y_1$$

or $$(1-x^2)\, y_2 - x\, y_1 = -m\left\{ \sqrt{1-x^2}\, y_1 \right\} = -m(-my) = m^2 y$$

$$(1-x^2)\,y_2 - x\,y_1 - m^2 y = 0$$

Now $\quad D^n\left[\,(1-x^2)\,y_2\,\right] - D^n\left[\,x\,y_1\,\right] - m^2 D^n\,[\,y\,] = 0$

Applying Leibnitz theorem to the product terms we have,

$$\left\{(1-x^2)\,y_{n+2} + n\cdot(-2x)\cdot y_{n+1} + \frac{n(n-1)}{1\cdot 2}(-2)\,y_n\right\}$$

$$-\left\{x\cdot y_{n+1} + n\cdot 1\cdot y_n\right\} - m^2 y_n = 0$$

ie., $\quad (1-x^2)\,y_{n+2} - 2\,n\,x\,y_{n+1} - n^2 y_n + ny_n - x\,y_{n+1} - ny_n - m^2 y_n = 0$

Thus $\quad (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - (m^2+n^2)\,y_n = 0$

---

20. *If $x = \sin t$ and $y = \cos m\,t$, prove that*

$$(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} + (m^2-n^2)\,y_n = 0$$

>> By data $\quad x = \sin t \quad$ and $\quad y = \cos mt$

$x = \sin t \;\Rightarrow\; t = \sin^{-1} x$ and $y = \cos mt$ becomes

$y = \cos(m\sin^{-1} x)$.

Differentiating *w.r.t x* we get

$$y_1 = -\sin(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}} \quad\text{or}\quad \sqrt{1-x^2}\,y_1 = -m\sin(m\sin^{-1}x)$$

Differentiating again *w.r.t x* we get,

$$\sqrt{1-x^2}\,y_2 + \frac{1}{2\sqrt{1-x^2}}(-2x)\,y_1 = -m\cos(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}}$$

or $\quad (1-x^2)\,y_2 - x\,y_1 = -m^2 y \qquad$ or $\quad (1-x^2)\,y_2 - x\,y_1 + m^2 y = 0$

Note *This result is almost same as that of the previous example and proceeding on the same lines we can arrive at the desired result.*

Thus $\quad (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} + (m^2-n^2)\,y_n = 0.$

---

21. *If $y^{1/m} + y^{-1/m} = 2x$, show that*

$$(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0$$

>> By data, $\quad y^{1/m} + y^{-1/m} = 2x$

ie., $\qquad y^{1/m} + \dfrac{1}{y^{1/m}} = 2x \qquad$ or $\qquad (y^{1/m})^2 + 1 = 2x\,(y^{1/m})$

ie., $(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$ which is a quadratic equation in $y^{1/m}$

If $t = y^{1/m}$ the equation assumes the form :

$$t^2 - 2xt + 1 = 0$$

We shall solve for t using the quadratic formula.

$$\therefore \quad t = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

ie., $$t = \frac{2x \pm 2\sqrt{x^2-1}}{2} \quad \text{or} \quad y^{1/m} = x \pm \sqrt{x^2-1}$$

Let $\quad y^{1/m} = x + \sqrt{x^2-1}$

$$\Rightarrow \quad y = \left[x + \sqrt{x^2-1} \right]^m$$

We now proceed to obtain a relation in $y, \; y_1, \; y_2$

Taking logarithms on both sides we get,

$$\log y = m \log \left[x + \sqrt{x^2-1} \right].$$

Differentiating $w.r.t\ x$ we get,

$$\frac{1}{y} y_1 = m \cdot \frac{1}{[x + \sqrt{x^2-1}]} \left\{ 1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x \right\}$$

ie., $$\frac{1}{y} y_1 = m \cdot \frac{1}{[x + \sqrt{x^2-1}]} \frac{[\sqrt{x^2-1} + x]}{\sqrt{x^2-1}} = \frac{m}{\sqrt{x^2-1}}$$

Also if, $y = [x - \sqrt{x^2-1}]^m$ we obtain

$$\frac{1}{y} y_1 = \frac{-m}{\sqrt{x^2-1}}$$

Thus $$\frac{1}{y} y_1 = \pm \frac{m}{\sqrt{x^2-1}}$$

Squaring and cross multiplying we get

$$(x^2-1) y_1^2 = m^2 y^2$$

Differentiating $w.r.t\ x$ again we get,

$$(x^2-1) 2y_1 y_2 + 2x y_1^2 = m^2 (2y y_1)$$

or $\quad (x^2-1) y_2 + x y_1 - m^2 y = 0$, on dividing by $2y_1$.

Now differentiating each term n times we have,

$$D^n\left[(x^2-1)y_2\right]+D^n[xy_1]-m^2D^n[y]=0$$

Applying Leibnitz theorem to the product terms we get,

$$\left\{(x^2-1)y_{n+2}+n\cdot 2x\cdot y_{n+1}+\frac{n(n-1)}{1\cdot 2}\cdot 2y_n\right\}$$
$$+\left\{x\cdot y_{n+1}+n\cdot 1\cdot y_n\right\}-m^2y_n=0$$

i.e., $(x^2-1)y_{n+2}+2nxy_{n+1}+n^2y_n-ny_n+xy_{n+1}+ny_n-m^2y_n=0$

Thus $\quad (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$

---

22. If $y=a(x+\sqrt{x^2-1})^n+b(x-\sqrt{x^2-1})^n$, prove that

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}=0.$$

>> By data, $y=a(x+\sqrt{x^2-1})^n+b(x-\sqrt{x^2-1})^n$

$$y_1=a\cdot n(x+\sqrt{x^2-1})^{n-1}\left\{1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right\}$$
$$+b\cdot n(x-\sqrt{x^2-1})^{n-1}\left\{1-\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right\}$$

i.e., $\quad y_1=an(x+\sqrt{x^2-1})^{n-1}\dfrac{(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}}+bn(x-\sqrt{x^2-1})^{n-1}\dfrac{(\sqrt{x^2-1}-x)}{\sqrt{x^2-1}}$

or $\quad \sqrt{x^2-1}\,y_1=an(x+\sqrt{x^2-1})^n-bn(x-\sqrt{x^2-1})^n$

Differentiating again *w.r.t* x and simplifying as before we get,

$$\sqrt{x^2-1}\cdot y_2+\frac{1}{2\sqrt{x^2-1}}\cdot 2xy_1$$
$$=\frac{an^2(x+\sqrt{x^2-1})^n+bn^2(x-\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$$

or $\quad (x^2-1)y_2+xy_1=n^2y$

Now, $\quad D^n[(x^2-1)y_2]+D^n[xy_1]=n^2y_n$

Applying Leibnitz theorem to each term in the L.H.S we get,

$$\left\{(x^2-1)y_{n+2}+n\cdot 2x\cdot y_{n+1}+\frac{n(n-1)}{1\cdot 2}\cdot 2y_n\right\}$$
$$+\left\{xy_{n+1}+n\cdot 1\cdot y_n\right\}-n^2y_n=0$$

ie, $\quad (x^2-1)\,y_{n+2}+2\,n\,x\,y_{n+1}+n^2 y_n-ny_n+x\,y_{n+1}+ny_n-n^2\,y_n=0$

Thus $\quad \boldsymbol{(x^2-1)\,y_{n+2}+(2\,n+1)\,x\,y_{n+1}=0}$

---

23. *If* $y=\sin^{-1}x$, *prove that*

$(1-x^2)\,y_{n+2}-(2\,n+1)\,x\,y_{n+1}-n^2\,y_n=0$. *Hence show that*

$$y_n(0)=\begin{cases}0 & \text{when } n \text{ is even}\\ (n-2)^2\,(n-4)^2\cdots 5^2\cdot 3^2\cdot 1^2 & \text{when } n \text{ is odd}\end{cases}$$

>> By data $y=\sin^{-1}x$

$$y_1=\frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad \sqrt{1-x^2}\,y_1=1.$$

Differentiating *w.r.t* x we get

$$\sqrt{1-x^2}\cdot y_2+\frac{1}{2\sqrt{1-x^2}}(-2x)\,y_1=0 \quad \text{or} \quad (1-x^2)\,y_2-x\,y_1=0.$$

Differentiating n times by applying Leibnitz theorem we get,

$$\left\{(1-x^2)\,y_{n+2}+n\cdot(-2x)\cdot y_{n+1}+\frac{n(n-1)}{1\cdot 2}(-2)\,y_n\right\}-\{x\,y_{n+1}+n\cdot 1\cdot y_n\}=0$$

or $\quad (1-x^2)\,y_{n+2}-2\,n\,x\,y_{n+1}-n^2\,y_n+ny_n-x\,y_{n+1}-ny_n=0$

Thus $\quad \boldsymbol{(1-x^2)\,y_{n+2}-(2\,n+1)\,x\,y_{n+1}-n^2\,y_n=0}$

Now putting $x=0$ we get $\quad y_{n+2}(0)=n^2\,y_n(0)$ ... (1)

If $\quad n=0: y_2(0)=0^2\,y_0(0)=0$

If $\quad n=2: y_4(0)=2^2\,y_2(0)=0$ etc.

Thus $\quad y_2(0)=0,\ y_4(0)=0,\ y_6(0)=0$

$\Rightarrow \quad$ **$y_n(0)=0$ when n is even.**

Also $\quad y_1=\dfrac{1}{\sqrt{1-x^2}}$ and $y_1(0)=1$

If $\quad n=1$ in (1) : $y_3(0)=1^2\,y_1(0)$

If $\quad n=3$ in (1) : $y_5(0)=3^2\,y_3(0)$

If $\quad n = 5$ in (1) : $y_7(0) = 5^2 y_5(0)$

$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$

$$y_{n-2}(0) = (n-4)^2 y_{n-4}(0)$$

$$y_n(0) = (n-2)^2 y_{n-2}(0)$$

Thus by back substitution we get when n is odd,

$$y_n(0) = (n-2)^2 (n-4)^2 \cdots 5^2 \cdot 3^2 \cdot 1^2 \text{ since } y_1(0) = 1$$

---

24. *If $x = \tan(\log y)$, find the value of*

$$(1+x^2) y_{n+1} + (2nx-1) y_n + n(n-1) y_{n-1}$$

>> By data $x = \tan(\log y) \Rightarrow \tan^{-1} x = \log y$ or $y = e^{\tan^{-1} x}$

Since the desired relation involves y_{n+1}, y_n and y_{n-1} we can find y_1 and differentiate n times the result associated with y_1 and y.

Consider $y = e^{\tan^{-1} x} \quad \therefore \quad y_1 = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$ or $(1+x^2) y_1 = y$

Differentiating n times we have

$$D^n[(1+x^2) y_1] = D^n[y]$$

Applying Leibnitz theorem onto L.H.S, we have,

$$\left\{ (1+x^2) D^n(y_1) + n \cdot 2x \cdot D^{n-1}(y_1) + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot D^{n-2}(y_1) \right\} = y_n$$

ie., $\quad (1+x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} - y_n = 0$

Thus $\quad (1+x^2) y_{n+1} + (2nx-1) y_n + n(n-1) y_{n-1} = 0$

---

25. *If $y = e^{x^2/2} \cos x$, show that*

$$y_{2n+2}(0) - 4n y_{2n}(0) + 2n(2n-1) y_{2n-2}(0) = 0$$

>> By data $y = e^{x^2/2} \cos x$

$\therefore \quad y_1 = e^{x^2/2}(-\sin x) + x e^{x^2/2} \cos x$

ie., $\quad y_1 = -e^{x^2/2} \sin x + xy \qquad \ldots (1)$

Differentiating again,

$$y_2 = -\left[e^{x^2/2}\cos x + x\,e^{x^2/2}\sin x\right] + \left[x\,y_1 + y\right]$$

ie., $$y_2 = -y - x\,e^{x^2/2}\sin x + x\,y_1 + y$$

ie., $$y_2 = x\,(y_1 - x\,y) + x\,y_1 \text{ by using (1).}$$

or $$y_2 - 2\,x\,y_1 + x^2\,y = 0 \qquad \ldots (2)$$

We have to now differentiate this relation $2\,n$ times and in respect of second and third terms we have to employ Leibnitz theorem in the form,

$$(uv)_{2n} = uv_{2n} + 2\,n_{C_1}\,u_1\,v_{2\,n-1} + 2\,n_{C_2}\,u_2\,v_{2\,n-2} + \cdots + u_{2n}\,v$$

From (2) we have $D^{2n}(y_2) - 2\,D^{2n}(x\,y_1) + D^{2n}(x^2\,y) = 0$

$$\therefore \quad y_{2\,n+2} - 2\left\{x\,D^{2n}(y_1) + 2\,n \cdot 1 \cdot D^{2\,n-1}(y_1)\right\}$$

$$+ \left\{x^2\,D^{2n}(y) + 2\,n \cdot 2\,x \cdot D^{2\,n-1}(y) + \frac{2\,n\,(2\,n-1)}{1 \cdot 2} \cdot 2 \cdot D^{2\,n-2}(y)\right\} = 0$$

ie., $$y_{2\,n+2} - 2\,x\,y_{2\,n+1} - 4\,n\,y_{2n} + x^2\,y_{2n} + 4n\,x\,y_{2\,n-1} + 2\,n\,(2\,n-1)\,y_{2\,n-2} = 0$$

Now putting $x = 0$ throughout we get,

$$\mathbf{y_{2\,n+2}(0) - 4n\,y_{2n}(0) + 2\,n\,(2\,n-1)\,y_{2\,n-2}(0) = 0}$$

---

26. *If $y = a\,x^{n+1} + \dfrac{b}{x^n}$ prove that $x^2\,y_2 = n\,(n+1)\,y$ & differentiate this result n times*

\>> By data, $y = a\,x^{n+1} + b\,x^{-n}$

$$\therefore \quad y_1 = a\,(n+1)\,x^n + b\,(-n)\,x^{-n-1}$$

$$y_2 = a\,(n+1)\,n\,x^{n-1} + b\,(-n)\,(-n-1)\,x^{-n-2}$$

or $$y_2 = a\,(n+1)\,n\,x^{n-1} + b\,n\,(n+1)\,x^{-n-2}$$

Now $$x^2\,y_2 = n\,(n+1)\left[a\,x^{n+1} + b\,x^{-n}\right]$$

ie., $$x^2\,y_2 = n\,(n+1)\,y$$

Differentiating this result n times we have

$$D^n(x^2\,y_2) - n\,(n+1)\,D^n(y) = 0$$

ie, $\left\{x^2 y_{n+2} + n \cdot 2x \cdot y_{n+1} + \dfrac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n\right\} - n(n+1) y_n = 0$

ie., $x^2 y_{n+2} + 2nx y_{n+1} + n^2 y_n - ny_n - n^2 y_n - ny_n = 0$

Thus $\mathbf{x^2 y_{n+2} + 2nx y_{n+1} - 2n y_n = 0}$

---

27. *If* $y = (ax+b)^m$ *show that if* $m > n$ $y_n = \dfrac{m!}{(m-n)!} a^n (ax+b)^{m-n}$

Interpret the result for the cases $m < n$ *and* $m = n$. *Hence show that*

$D^{2n}[(x^2+1)^n] = 2n!$

>> By data $y = (ax+b)^m$ where $m > n$

Differentiating successively we obtain,

$$y_n = m(m-1)(m-2)\cdots[m-(n-1)]a^n(ax+b)^{m-n} \quad \ldots (\textit{Result } F_3)$$

Multiplying and dividing by the term $(m-n)!$ which is given by

$(m-n)(m-n-1)(m-n-2)\cdots 3 \cdot 2 \cdot 1$

[$\because$ $m > n \Rightarrow (m-n) > 0$ *and if* $(m-n) = k$, *a positive integer then*

$k! = k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1$]

Thus $y_n = \{m(m-1)(m-2)\cdots m-(n-1)\}$

$$\times \frac{\{(m-n)(m-n-1)\cdots 3 \cdot 2 \cdot 1\}}{(m-n)!} \cdot a^n \cdot (ax+b)^{m-n}$$

Combining all the terms in the numerator, the same represents the product of first m natural numbers which is $m!$

Thus if $m > n$ $\quad \boldsymbol{y_n = \dfrac{m!}{(m-n)!} a^n (ax+b)^{m-n}}$ ... (1)

Next if $m = n$ $\quad D^n\left[(ax+b)^m\right] = \dfrac{n!}{0!} a^n (ax+b)^0$

$\therefore \quad y_n = n!\,a^n$ = constant, when $m = n$

Thus if $m = n$, $\quad \boldsymbol{D^n\left[(ax+b)^n\right] = n!\,a^n}$

Also if $m < n$ then $D^n\left[(ax+b)^m\right] = D^n(\text{constant}) = 0$

$\therefore \quad D^{n-m}\left[D^m(ax+b)^m\right] = D^k(\text{constant}) = 0$ since $k = n - m > 0$

Thus if $m < n$ $\quad \boldsymbol{D^n\left[(ax+b)^m\right] = 0}$

Also we have to find $D^{2n}\left[\,(x^2+1)^n\,\right]$ using the above results

Expanding $(x^2+1)^n$ by the binomial theorem we obtain,

$$(x^2+1)^n = (x^2)^n + n_{C_1}(x^2)^{n-1}\cdot 1 + n_{C_2}(x^2)^{n-2}\cdot 1^2 + \cdots + 1$$

ie., $$(x^2+1)^n = x^{2n} + n_{C_1}x^{2n-2} + n_{C_2}x^{2n-4} + \cdots + 1$$

$$\therefore \quad D^{2n}\left[\,(x^2+1)^n\,\right] = D^{2n}(x^{2n}) + n_{C_1}D^{2n}(x^{2n-2})$$

$$+\, n_{C_2}D^{2n}(x^{2n-4}) + \cdots + 0 \qquad \ldots (4)$$

We have (2) and (3) when $a = 1, b = 0$ in the form

$$D^n(x^n) = n! \text{ and } D^n(x^m) = 0 \text{ if } m < n$$

Using these results in (4) we get,

$$D^{2n}\left[\,(x^2+1)^n\,\right] = 2n! + n_{C_1}\cdot 0 + n_{C_2}\cdot 0 + \cdots = 2n!$$

Thus $\mathbf{D^{2n}\left[\,(x^2+1)^n\,\right] = 2\,n!}$

---

28. *If $x + y = 1$ then show that,*

$$\frac{d^n}{dx^n}\left[\,x^n y^n\,\right] = n!\left\{ y^n - (n_{C_1})^2 y^{n-1}x + (n_{C_2})^2 y^{n-2}x^2 + \cdots + (-1)^n x^n \right\}$$

>> By data $x + y = 1$, $\therefore\ y = 1 - x$

Let us take $u = x^n$ and $v = y^n = (1-x)^n$

We shall find $u_1, u_2, \cdots u_{n-2}, u_{n-1}\,u_n$ and $v_1, v_2, \cdots v_{n-2}, v_{n-1}, v_n$ with the aim to apply Leibnitz theorem for the product $u\,v$.

Consider $u = x^n$ and its successive derivatives are as follows

$$u_1 = n\,x^{n-1}\ ;\ u_2 = n(n-1)x^{n-2} \text{ etc.}$$

$$u_{n-2} = n(n-1)(n-2)\cdots 3x^2 = \frac{n!}{2}x^2$$

$$u_{n-1} = n(n-1)(n-2)\cdots 3\cdot 2x = n!\,x$$

$$u_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

Now consider $v = y^n$ where $y = 1 - x$.

$$v_1 = n\,y^{n-1} \cdot y' = n\,y^{n-1}(-1), \quad v_2 = n(n-1)\,y^{n-2}(-1)^2 \text{ etc.}$$

$$v_{n-2} = \frac{n!}{2} y^2 (-1)^{n-2}, \quad v_{n-1} = n!\,y(-1)^{n-1}, \quad v_n = n!\,(-1)^n$$

We have Leibnitz theorem,

$$D^n(uv) = uv_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2\, u_2 v_{n-2}$$

$$+ \cdots + {}^nC_{n-2}\, u_{n-2} v_2 + {}^nC_{n-1}\, u_{n-1} v_1 + u_n v$$

$$\therefore \quad D^n(x^n y^n) = x^n \cdot n!\,(-1)^n + {}^nC_1 \cdot n x^{n-1} \cdot n!\,y(-1)^{n-1}$$

$$+ {}^nC_2 \cdot n(n-1)\,x^{n-2} \cdot \frac{n!}{2} y^2 (-1)^{n-2}$$

$$+ \cdots + {}^nC_{n-2} \cdot \frac{n!}{2} x^2 \cdot n(n-1)\,y^{n-2}(-1)^2 + {}^nC_{n-1} \cdot n!\,x \cdot (-ny^{n-1}) + n!\,y^n$$

We know that $n = {}^nC_1 = {}^nC_{n-1}\ ; \ \dfrac{n(n-1)}{2} = {}^nC_2 = {}^nC_{n-2}$ etc.

$$\therefore \quad D^n(x^n y^n) = n!\Big\{(-1)^n x^n + ({}^nC_1)^2(-1)^{n-1} x^{n-1} y$$

$$+ ({}^nC_2)^2(-1)^{n-2} x^{n-2} y^2 + \cdots + ({}^nC_{n-2})^2(-1)^2 x^2 y^{n-2}$$

$$+ ({}^nC_{n-1})^2(-1)^1 x y^{n-1} + y^n\Big\}$$

Reversing the order of the terms in R.H.S we have

$$D^n(x^n y^n) = n!\Big\{y^n - ({}^nC_1)^2 x y^{n-1} + ({}^nC_2)^2 x^2 y^{n-2} \cdots$$

$$+ ({}^nC_{n-1})^2(-1)^{n-1} x^{n-1} y + (-1)^n x^n\Big\}$$

Remark : *The problem can also be worked by the method of mathematical induction*

29. *If* $y = e^{-x^2}$, *show that* $y_{n+1} + 2x\,y_n + 2n\,y_{n-1} = 0$

>> $\quad y = e^{-x^2}$

$$y_1 = e^{-x^2}(-2x) \text{ or } y_1 = -2xy$$

We have, $y_1 + 2xy = 0$

Differentiating this result n times we have,

$$D^n [y_1] + 2 D^n [xy] = 0$$

ie.,
$$y_{n+1} + 2 \left\{ x y_n + n \cdot 1 \cdot y_{n-1} \right\} = 0$$

Thus
$$y_{n+1} + 2 x y_n + 2 n y_{n-1} = 0$$

>> $y = \sinh (m \sinh^{-1} x)$

$\therefore$
$$y_1 = \cosh (m \sinh^{-1} x) \cdot \frac{m}{\sqrt{1+x^2}}$$

or
$$\sqrt{1+x^2}\, y_1 = m \cosh (m \sinh^{-1} x)$$

Differentiating w.r.t x again we get,

$$\sqrt{1+x^2}\, y_2 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x y_1 = m \sinh (m \sinh^{-1} x) \cdot \frac{m}{\sqrt{1+x^2}}$$

or
$$(1 + x^2) y_2 + x y_1 = m^2 y$$

Differentiating each term n times we have,

$$D^n [(1 + x^2) y_2] + D^n [x y_1] - m^2 D^n (y) = 0$$

Applying Leibnitz theorem to the product terms we get,

$$\left\{ (1 + x^2) y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n (n - 1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\} + \left\{ x y_{n+1} + n \cdot 1\, y_n \right\} - m^2 y_n = 0$$

Thus $(1 - x^2) y_{n+2} + (2n + 1) x y_{n+1} + (n^2 - m^2) y_n = 0$ **is the required relation.**

EXERCISES

Find y_n *for the following functions.* [1 and 2]

1. $y = x \cos 3x \cos 5x$ 2. $y = x^2 e^x \cos x$

3. If $y = x^2 e^x$, prove that $y_n = (x^2 + 2 n x + n^2 - n) e^x$

4. If $y = x \log \left(\frac{x-1}{x+1} \right)$, show that $D^n y = (- 1)^n (n - 2) ! \left[\frac{x - n}{(x - 1)^n} - \frac{x + n}{(x + 1)^n} \right]$

5. If $y = (\sin^{-1} x)^2$ prove that $(1-x^2)\, y_2 - x\, y_1 = 2$
and then apply Leibnitz theorem to differentiate the result n times

6. If $y = \sin(m \sin^{-1} x)$ prove that

(a) $(1-x^2)\, y_2 - x\, y_1 + m^2 y = 0$

(b) $(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$

7. If $y = \cos \log(x^2 - 2x + 1)$, prove that

$$(x-1)^2 y_{n+2} + (2n+1)(x-1)\, y_{n+1} + (n^2+4)\, y_n = 0$$

8. If $y = \left[x + \sqrt{1+x^2}\right]^m$ show that,

$$(1+x^2)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

9. If $y = \left[\log\left(x + \sqrt{x^2+a^2}\right)\right]^2$, prove that

$(1+x^2)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + n^2 y_n = 0$ Hence deduce that .

$$y_n(0) = \begin{cases} 0, & \text{when } n \text{ is even} \\ (-1)^n \cdot 1^2 \cdot 2^2 \cdot 3^2 \cdots n^2, & \text{when } n \text{ is odd} \end{cases}$$

10. If $y = a \cos h(\log x^m) + b \sin h(\log x^m)$ show that

$$x^2 y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

ANSWERS

1. $$\frac{1}{2}\left[x \cdot 8^n \cos\left(\frac{n\pi}{2} + 8x\right) + n \cdot 8^{n-1} \cos\left((n-1)\frac{\pi}{2} + 8x\right)\right]$$
$$+ \frac{1}{2}\left[x \cdot 2^n \cos\left(\frac{n\pi}{2} + 2x\right) + n \cdot 2^{n-1} \cos\left((n-1)\frac{\pi}{2} + 2x\right)\right]$$

2. $$2^{n/2} e^x \left[x^2 \cos\left(n\frac{\pi}{4} + x\right) + \sqrt{2}\, n\, x \cos\left((n-1)\frac{\pi}{4} + x\right)\right.$$
$$\left. + \frac{n(n-1)}{2} \cos\left((n-2)\frac{\pi}{4} + x\right)\right]$$

5. $$(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0$$

Rolle's theorem and mean value theorems

1.1 Continuity and Diff...

A function $f(x)$ is said to be *continuous* at a point $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$

where $f(a)$ means, the value of $f(x)$ at $x = a$ and it should not be infinite, indeterminate, imaginary.

A function $f(x)$ is said to be *differentiable* at a point $x = a$ if $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is unique.

A function $f(x)$ is said to be continuous or differentiable in an interval if it is continuous or differentiable at each point of the interval. In simple words, we can say that $f(x)$ is continuous in an interval if the graph of $f(x)$ do not have any breaks in that interval. Differentiability accounts for the smoothness of the curve

...otes and No...

1. *If a function is differentiable then it is necessarily continuous but not conversely That is, differentiability implies continuity.*

2. *If $f(x)$ and $g(x)$ are two continuous functions then $k f(x)$, $k g(x)$ [where k is a constant], $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $f(x) / g(x)$ where $g(x) \neq 0$ for all x are all continuous functions.*

3. *Common functions like constant, any polynomial, $\sin x$, $\cos x$, e^x, $\sinh x$, $\cosh x$ etc. are continuous at all points. Also we can say that, $\log x$ is not continuous at $x = 0$, $1/(x - 1)$ is not continuous at $x = 1$, $\tan x$ is not continuous at $x = \pi/2$ etc.*

1 $[a, b]$: *Closed interval a, b. It includes all the points between a & b including a & b*

2 (a, b) : *Open interval a, b. It includes all the points between a & b excluding a & b*

ie., $x \in [a, b] \Rightarrow a \leq x \leq b$; $x \in (a, b) \Rightarrow a < x < b$

1.22 Rolle's theorem and its geometrical interpretation

Statement

such that

Geometrical interpretation (meaning)

Geometrically the three conditions can be interpreted as follows

(i) The curve $f(x)$ in the interval a, b should not have any breaks including the end points.

(ii) The curve $f(x)$ is smooth and it must possess tangent at all points in the interval a, b except at the end points where tangents cannot be drawn

(iii) The ordinates $f(a)$, $f(b)$ respectively corresponding to a, b must be of the same height from the x-axis

When all these conditions are satisfied, geometrically the theorem means that there exists atleast one point strictly between the end points at which the tangent is parallel to the x-axis.

A picturesque illustration is as follows.

In the above figure the points P and Q correspond to $x = a$, $x = b$ and the corresponding heights $f(a)$, $f(b)$ are equal It can be seen from the figure that there are two points L and M on the curve $[a < c_1 < b,\ a < c_2 < b]$ at which the tangents are parallel to the x-axis.

Remark *This theorem is the foundation for all other theorems to follow*

We now proceed to establish mean value theorems by applying Rolle's theorem

1.23

Proof : Let us define a new function

$$\phi(x) = f(x) - k.x \qquad \ldots(1)$$

where k is a constant to be chosen suitably later. Since $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and that kx is also continuous in $[a, b]$, differentiable in (a, b) we can conclude that $\phi(x)$ is also continuous in $[a, b]$, differentiable in (a, b)

From (1) we have, $\phi(a) = f(a) - k \cdot a$; $\phi(b) = f(b) - k \cdot b$

$\therefore$ $\phi(a) = \phi(b)$ holds good if

$$f(a) - k \cdot a = f(b) - k \cdot b \quad \text{or} \quad k(b-a) = f(b) - f(a)$$

$$k = \frac{f(b) - f(a)}{b - a} \qquad \ldots(2)$$

Hence if k is chosen as given by (2), then $\phi(x)$ satisfy all the conditions of Rolle's theorem. Therefore by Rolle's theorem there exists atleast one point c in (a, b) such that $\phi'(c) = 0$

Differentiating (1) w.r.t. x we have, $\phi'(x) = f'(x) - k$
and $\phi'(c) = 0$ yields $f'(c) - k = 0$
ie., $k = f'(c)$ ... (3)

Equating the R.H.S of (2) and (3) we have

$$\boxed{\frac{f(b) - f(a)}{b - a} = f'(c)} \qquad \ldots(4)$$

This proves Lagrange's mean value theorem.

1. *The theorem can also be put in the following forms :*

$$f(b) - f(a) = (b - a) f'(c) \quad \text{or} \quad f(b) = f(a) + (b - a) f'(c) \qquad (5)$$

2. *Further if the length of the interval $[a, b]$ is h we have $b - a = h$ or $b = a + h$. Also if we set $\theta = \dfrac{c - a}{h} = \dfrac{c - a}{b - a}$ we observe that θ lies between 0 and 1. That is $0 < \theta < 1$. Now $c = a + \theta h$ and $b = a + h$. The theorem in the form (5) becomes*

$$f(a + h) = f(a) + h f'(a + \theta h) \qquad \ldots(6)$$

Let $P = (a, f(a))$, $Q = (b, f(b))$ be any two points on the curve representing $y = f(x)$.

$$\text{slope of } PQ = \frac{f(b) - f(a)}{b-a}$$

As per the conditions of the theorem, the curve $f(x)$ has no breaks in the interval including the end points and possesses tangents at all points within the interval.

Geometrically the theorem means that there exists atleast one point on the curve at which the tangent is parallel to the line joining the end points. In the above figure there are two points L and M at which the tangents are parallel to the line PQ.

Proof . Let us define a new function

$$\phi(x) = f(x) - k\,g(x) \qquad \ldots (1)$$

where k is a constant to be chosen suitably later. From the given conditions it is evident that $\phi(x)$ is also continuous in $[a, b]$, differentiable in (a, b).

Further from (1) we have,

$$\phi(a) = f(a) - k\,g(a) \;;\; \phi(b) = f(b) - k\,g(b)$$

$$\phi(a) = \phi(b) \quad \text{holds good if}$$

$$f(a) - k\,g(a) = f(b) - k\,g(b)$$

ie., $$k\,[g(b) - g(a)] = f(b) - f(a)$$

$$k = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Here $g(b) \neq g(a)$. Because if $g(b) = g(a)$ then $g(x)$ would satisfy all the conditions of Rolle's theorem and accordingly there must exist atleast one point c in

(a, b) such that $g'(c) = 0$ This contradicts the data that $g'(x) \neq 0$ for all x in (a, b)

Hence if k is chosen as given by (2) then $\phi(x)$ satisfy all the conditions of Rolle's theorem Therefore by Rolle's theorem there exists atleast one value c in (a, b) such that $\phi'(c) = 0$

Differentiating (1) w.r.t. x we have,

$$\phi'(x) = f'(x) - kg'(x) \text{ and } \phi'(c) = 0 \text{ yields}$$

$$f'(c) - kg'(c) = 0 \quad \text{ie. } f'(c) = kg'(c)$$

$$\therefore \quad k = \frac{f'(c)}{g'(c)}$$

Equating the R.H.S of (2) and (3) we have

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

This proves Cauchy's mean value theorem

1. *We can deduce Lagrange's mean value theorem from Cauchy's mean value theorem*

Taking $g(x) = x$ *we have* $g(a) = a,\ g(b) = b$

Also $g'(x) = 1 \Rightarrow g'(c) = 1$

Hence Cauchy's mean value theorem becomes,

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1} \quad \text{ie.,} \quad \frac{f(b)-f(a)}{b-a} = f'(c)$$

This is Lagrange's mean value theorem

2. *We can also deduce Cauchy's mean value theorem from Lagrange's theorem*

Let us take the format of Lagrange's mean value theorem for $f(x)$ *and* $g(x)$ *in the form*

$$\frac{f(b)-f(a)}{b-a} = f'(c_1) \quad \ldots (1)$$

and

$$\frac{g(b)-g(a)}{b-a} = g'(c_2) \quad \ldots (2)$$

Dividing (1) by (2) we have,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c_1)}{g'(c_2)}$$

This is Cauchy's mean value theorem provided $c_1 = c_2 = c$ *where* $a < c < b$

2.5 A general mean value theorem or Taylor's theorem for a function of one variable

Statement *If a function $f(x)$ and its first $(n-1)$ derivatives are continuous in $[a, b]$ and $f^{(n)}(x)$ exists in (a, b) then there exists atleast one point c in (a, b) such*

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots$$

Note : 1. *Taking $b - a = h$ and $c = a + \theta h$ where $\theta = \frac{c-a}{b-a}$ we observe that $0 < \theta < 1$. Taylor's theorem becomes*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h)$$

2. *By taking $n = 1$ in both the forms of Taylor's theorem we get*

$$f(b) = f(a) + (b-a) f'(c) \quad or \quad f(a+h) = f(a) + hf'(a+\theta h)$$

These are the two forms of Lagrange's mean value theorem. Hence the same is referred to as Lagrange's first mean value theorem.

When $n = 2$ we get

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2} f''(c)$$

This is referred to as Lagrange's second mean value theorem and so on.

Verification of Rolle's theorem, Lagrange's and Cauchy's mean value theorems

Working procedure for problems

- We ensure the continuity of the given function / functions in the interval. $f(a) = f(b)$ also has to be ensured for Rolle's theorem.
- The derivative of the given function / functions must be found and we ensure differentiability in the interval.
- We need to identify the value / values of c belonging to the open interval satisfying the relevant theorem.

WORKED PROBLEMS

$f(x) = x^2$ is continuous in $[-1, 1]$ & $f'(x) = 2x$ exist for all values in $(-1, 1)$ $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$ $\quad\therefore\quad$ $f(-1) = f(1)$

Hence all the three conditions of the theorem are satisfied.

Now consider $f'(c) = 0$ that is $2c = 0$ or $c = 0$
$\mathbf{c = 0 \in (-1, 1)}$ and hence Rolle's theorem is verified

Geometrically $f(x) = y = x^2$ is a parabola symmetrical about the y-axis passing through the origin and that the x-axis itself is the tangent to the curve at $x = c = 0$

Verify Rolle's theorem for the following functions

32. $f(x) = x^2(1-2x+x^2) = x^2 - 2x^3 + x^4$

$f(x)$ is continuous in $[0, 1]$

$f'(x) = 2x - 6x^2 + 4x^3$ exists in $(0, 1)$

Also $f(0) = 0 = f(1)$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

ie., $2c - 6c^2 + 4c^3 = 0$ *or* $2c(1 - 3c + 2c^2) = 0$

ie., $2c(2c-1)(c-1) = 0 \Rightarrow 2c = 0,\ 2c - 1 = 0,\ c - 1 = 0$

$c = 0 \quad c = 1/2 \quad c = 1$

$\mathbf{c = 1/2 \in (0, 1)}$. **Thus Rolle's theorem is verified**

33. $f(x) = (x^2 + 2x)\,e^{-x/2}$ is continuous in $[-2, 0]$

$f'(x) = (x^2 + 2x)\,e^{-x/2}(-1/2) + (2x + 2)\,e^{-x/2}$

i.e., $\quad f'(x) = \dfrac{e^{-x/2}}{2}(-x^2 - 2x + 4x + 4)$

$\therefore \quad f'(x) = \dfrac{e^{-x/2}}{2}(-x^2 + 2x + 4)$

$f'(x)$ exists in $(-2, 0)$. Also $f(-2) = 0 = f(0)$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

From (1) we have $\dfrac{e^{-c/2}}{2}(-c^2 + 2c + 4) = 0$

Since $e^{-c/2}$ cannot be zero, we must have

$$-c^2 + 2c + 4 = 0 \quad \text{or} \quad c^2 - 2c - 4 = 0$$

$$c = \frac{2 \pm \sqrt{4+16}}{2} = \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

$c = 1 - \sqrt{5} \approx -1.236 \in (-2, 0)$ **Thus Rolle's theorem is verified.**

34. $\quad f(x) = (x-a)^p (x-b)^q$ is continuous in $[a, b]$

$$f'(x) = (x-a)^p q (x-b)^{q-1} + p(x-a)^{p-1}(x-b)^q$$

$$= (x-a)^{p-1}(x-b)^{q-1}[q(x-a) + p(x-b)]$$

$$f'(x) = (x-a)^{p-1}(x-b)^{q-1}[(q+p)x - (qa+pb)] \qquad \ldots (1)$$

$f'(x)$ exists in (a, b)

Also $\quad f(a) = 0 = f(b)$

Hence all the conditions of the theorem are satisfied.

Now Consider $f'(c) = 0$

From (1), $(c-a)^{p-1}(c-b)^{q-1}[(q+p)c - (qa+pb)] = 0$

$$\Rightarrow \quad c - a = 0, \ c - b = 0, \ (q+p)c - (qa+pb) = 0$$

i.e., $\quad c = a, \ c = b, \ c = \dfrac{pb+qa}{p+q}$

a, b are the end points. $c = pb + qa / p + q$ is the x coordinate of the point which divides the line joining $[a, f(a)]$, $[b, f(b)]$ internally in the ratio $p : q$

$\therefore \quad c = pb + qa / p + q \in (a, b)$ **Thus Rolle's theorem is verified.**

35. $f(x) = e^x(\sin x - \cos x)$ is continuous in $[\pi/4, 5\pi/4]$

$$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x)$$

$$f'(x) = 2e^x \sin x \qquad \ldots (1)$$

$f(x)$ is differentiable in $(\pi/4, 5\pi/4)$

$$f(\pi/4) = e^{\pi/4}(\sin \pi/4 - \cos \pi/4) = e^{\pi/4}(1/\sqrt{2} - 1/\sqrt{2}) = 0$$

$$f(5\pi/4) = e^{5\pi/4}(\sin 5\pi/4 - \cos 5\pi/4)$$
$$= e^{5\pi/4}(-1/\sqrt{2} + 1/\sqrt{2}) = 0$$

$$f(\pi/4) = 0 = f(5\pi/4)$$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

From (1) we have, $2e^c \sin c = 0$. But $e^c \neq 0$

$$\sin c = 0 \Rightarrow c = n\pi \quad \text{where } n = 0, 1, 2, \ldots$$

But $c = \pi \in (\pi/4, 5\pi/4)$ **Thus Rolle's theorem is verified.**

36. $f(x) = \dfrac{\sin 2x}{e^{2x}}$ is continuous in $[0, \pi/2]$

$$f'(x) = \frac{e^{2x} \cdot 2\cos 2x - \sin 2x \cdot 2e^{2x}}{(e^{2x})^2} = \frac{2e^{2x}(\cos 2x - \sin 2x)}{(e^{2x})^2}$$

ie., $$f'(x) = \frac{2(\cos 2x - \sin 2x)}{e^{2x}} \qquad \ldots (1)$$

$f'(x)$ exists in $(0, \pi/2)$

Also $f(0) = 0 = f(\pi/2)$ $\because$ $\sin 0 = 0 = \sin \pi$

Hence all the conditions of the theorem are satisfied.

Now consider $f'(c) = 0$

We have from (1)

$$\frac{2(\cos 2c - \sin 2c)}{e^{2c}} = 0$$

$\Rightarrow$ $\cos 2c - \sin 2c = 0$

ie., $\cos 2c = \sin 2c$ or $\tan 2c = 1 \Rightarrow 2c = \pi/4$

$c = \pi/8 \in (0, \pi/2)$. **Thus Rolle's theorem is verified.**

Rolle's theorem for the fn

>> The given $f(x)$ is continuous in $[a, b]$ since $0 < a < b$

The given $f(x)$ can be written in the form

$$f(x) = \log(x^2 + ab) - \log(a+b) - \log x$$

$$f'(x) = \frac{2x}{x^2+ab} - 0 - \frac{1}{x} = \frac{2x^2 - (x^2+ab)}{(x^2+ab)x}$$

ie., $$f'(x) = \frac{x^2 - ab}{(x^2+ab)x} \qquad \ldots (1)$$

$f'(x)$ exists in (a, b)

Also $$f(a) = \log\left[\frac{a^2+ab}{(a+b)a}\right] = \log 1 = 0$$

$$f(b) = \log\left[\frac{b^2+ab}{(a+b)b}\right] = \log 1 = 0$$

$$f(a) = f(b)$$

Hence all the conditions of the theorem are satisfied

Now consider $f'(c) = 0$

From (1) we have, $\dfrac{c^2 - ab}{(c^2+ab)c} = 0 \Rightarrow c^2 - ab = 0$ ie., $c = \pm\sqrt{ab}$

$c = +\sqrt{ab} \in (a, b)$ and we know that $\sqrt{ab}$ is the geometric mean of a and b

Remark : *Consider the function $f(x) = x^3$ in $[-1, 1]$. $f(x)$ is continuous in $[-1, 1]$, $f'(x) = 3x^2$ exists in $(-1, 1)$, $f(-1) \neq f(1)$. But $f'(c) = 0$ gives $3c^2 = 0$ or $c = 0 \in (-1, 1)$ Though one of the conditions of the theorem is not satisfied we have a value $c = 0 \in (-1, 1)$ such that $f'(c) = 0$ Such situations are interpreted as follows.*

When all the conditions of the theorem are satisfied definitely we will have atleast one value 'c' within the interval such that $f'(c) = 0$

ge's mean value theorem for the following functions

$f(x) = \log x$ in $[1, e]$

38. We have Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Here $f(x) = \log x\ ; a = 1,\ b = e$

$f(x)$ is continuous in $[1, e]$ Also $f'(x) = 1/x$

$f(x)$ is differentiable in $(1, e)$

Hence the theorem becomes

$$\frac{f(e)-f(1)}{e-1} = \frac{1}{c} \quad \textit{ie.,} \quad \frac{\log e - \log 1}{e-1} = \frac{1}{c}$$

But $\log 1 = 0,\ \log e = 1$ and hence we have $\dfrac{1}{e-1} = \dfrac{1}{c}$

or $c = e - 1 \approx 2.7 - 1 = 1.7 \in (1, e)$ **Thus the theorem is verified.**

39. We have Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

The given $f(x)$ is continuous in $[0, 4]$

$f(x) = (x-1)(x-2)(x-3)\ ;\ a = 0,\ b = 4$ by data.

$\therefore$ $f(b) = f(4) = 3\,.\,2\,.\,1 = 6$ and

$f(a) = f(0) = (-1)(-2)(-3) = -6$

We have $f(x) = x^3 - 6x^2 + 11x - 6$ in the simplified form.

$f'(x) = 3x^2 - 12x + 11$ exists in $(0, 4)$

The theorem becomes

$$\frac{f(4)-f(0)}{4-0} = 3c^2 - 12c + 11$$

ie., $\dfrac{6-(-6)}{4} = 3c^2 - 12c + 11$

or $3c^2 - 12c + 8 = 0$

$c = \dfrac{12 \pm \sqrt{144 - 96}}{6} = \dfrac{12 \pm \sqrt{48}}{6}$; $c = \dfrac{12 + \sqrt{48}}{6}, \dfrac{12 - \sqrt{48}}{6}$

$c \approx 3.15$ **and 0.85 both belong to** $(0,\ 4)$ **Thus the theorem is verified.**

40. We have the theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

$f(x) = x^3 - 3x^2 + 2x$. $f(x)$ is continuous in $[0,\ 1/2]$

$f'(x) = 3x^2 - 6x + 2$ $\therefore$ $f(x)$ is differentiable in $(0,\ 1/2)$

With $a = 0$, $b = 1/2$ and $f(x) = x(x-1)(x-2)$ the theorem becomes

$\dfrac{1/2(1/2-1)(1/2-2)-0}{1/2-0} = 3c^2 - 6c + 2$ or

$(-1/2)(-3/2) = 3c^2 - 6c + 2$

ie., $12c^2 - 24c + 5 = 0$ $\therefore$ $c = \dfrac{24 \pm \sqrt{576 - 240}}{24} = \dfrac{24 \pm \sqrt{336}}{24}$

ie., $c \approx 1.76$ and 0.24

Here $c = 0.24 \in (0,\ 1/2)$ **Thus the theorem is verified.**

41. We have the theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

$f(x) = \cos^2 x$ is continuous in $[0,\ \pi/2]$

$f'(x) = -2\cos x \sin x = -\sin 2x$ exists in $(0,\ \pi/2)$

$f(b) = f(\pi/2) = \cos^2(\pi/2) = 0$; $f(a) = f(0) = \cos^2 0 = 1$

$\therefore$ the theorem becomes

$\dfrac{0-1}{\pi/2-0} = -\sin 2c$ or $\sin 2c = 2/\pi \Rightarrow 2c = \sin^{-1}(2/\pi)$

$c = (1/2)\cdot\sin^{-1}(2/\pi) \approx 0\cdot 345$; But $\pi/2 = 1\cdot 57$

Here $c = 0\cdot 345 \in (0,\ 1\cdot 57)$ **Thus the theorem is verified.**

42. We have the theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

$f(x) = \tan^{-1} x$ is continuous in $[0, 1]$

$f'(x) = 1/1+x^2$ exists in $(0, 1)$

$f(b) = f(1) = \tan^{-1}(1) = \pi/4$; $f(a) = f(0) = \tan^{-1} 0 = 0$

the theorem becomes

$$\frac{\pi/4-0}{1} = \frac{1}{1+c^2} \quad \text{or} \quad \pi(1+c^2) = 4$$

ie, $\pi c^2 = 4-\pi$ or $c^2 = (4-\pi)/\pi$

$c = \sqrt{(4-\pi)/\pi}$

$c \approx 0 \cdot 523 \in (0, 1)$. **Thus the theorem is verified.**

43. *Find θ of Lagrange's mean [illegible] theorem for the function $f(x) = e^x$ in $(0, 1)$*

>> We have the theorem in the form

$$f(b) = f(a) + (b-a) f'(a+\theta h) \qquad \dots (1)$$

$f(x) = e^x$ is continuous in $[0, 1]$ $f'(x) = e^x$ exists in $(0, 1)$

Here $a = 0,\ b = 1 \quad \therefore\ f'(a+\theta h) = f'(0 h) = e^{\theta h}$

$f(b) = f(1) = e^1 = e$; $f(a) = f(0) = e^0 = 1$

Hence (1) becomes,

$$e = 1 + (1-0)\, e^{\theta h} \quad \text{i.e.,} \quad e - 1 = e^{\theta h}$$

But $h = b-a = 1 \quad \therefore \quad e^{\theta} = e-1 \Rightarrow \theta = \log(e-1)$

i.e., $\theta = \log(e-1) \approx \log_e(2.7-1) = \log_e(1.7) \approx 0.53 < 1$

Thus the required $\theta = 0.53$

44. [illegible] functions

>> [illegible] $x+9$ and $\sqrt{x}$ [illegible] in $[0, 16]$

[illegible]

44. We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \text{Here } a = 0,\ b = 16$$

Let $\quad f(x) = \sqrt{x+9} \quad g(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x+9}} \qquad g'(x) = \frac{1}{2\sqrt{x}}$$

$f(x)$ and $g(x)$ are continuous in $[0, 16]$, differentiable in $(0, 16)$
$g'(x) \neq 0 \quad \forall\ x \in (0, 16)$

Hence the theorem becomes

$$\frac{f(16)-f(0)}{g(16)-g(0)} = \frac{1/2\sqrt{c+9}}{1/2\sqrt{c}}$$

ie., $$\frac{\sqrt{25}-\sqrt{9}}{\sqrt{16}-\sqrt{0}} = \frac{\sqrt{c}}{\sqrt{c+9}} \quad ie., \quad \frac{5-3}{4} = \frac{\sqrt{c}}{\sqrt{c+9}}$$

ie., $$\frac{1}{2} = \frac{\sqrt{c}}{\sqrt{c+9}} \qquad \therefore \quad 4c = c+9 \text{ or } c = 3$$

The value $c = 3 \in (0, 16)$. **Thus the theorem is verified.**

45. We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \text{where } a = 3,\ b = 7 \text{ by data.}$$

$f(x) = e^x$ gives $f'(x) = e^x$; $g(x) = e^{-x}$ gives $g'(x) = -e^{-x}$

$f(x)$, $g(x)$ are continuous in $[3, 7]$, differentiable in $(3, 7)$
$g'(x) \neq 0 \quad \forall\ x \in (3, 7)$

Hence the theorem becomes

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{e^c}{-e^{-c}}$$

ie., $$\frac{e^7-e^3}{\dfrac{1}{e^7}-\dfrac{1}{e^3}} = -e^{2c} \quad \text{or} \quad \frac{(e^7-e^3)e^{10}}{(e^3-e^7)} = -e^{2c}$$

ie., $$\frac{(e^3-e^7)e^{10}}{(e^3-e^7)} = e^{2c} \Rightarrow e^{2c} = e^{10} \Rightarrow 2c = 10 \text{ or } c = 5$$

$c = 5 \in (3, 7)$. **Thus the theorem is verified.**

46. $f(x) = \log x$. Let $g(x) = f'(x) = 1/x$

$f'(x) = 1/x$; $g'(x) = -1/x^2$ Also $a = 1$, $b = e$

$f(x)$ and $g(x)$ are continuous in $[1, e]$, differentiable in $(1, e)$

$g'(x) \neq 0 \quad \forall \ x \in (1, e)$

We have Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

ie.,
$$\frac{f(e)-f(1)}{g(e)-g(1)} = \frac{1/c}{-1/c^2}$$

ie., $\dfrac{\log e - \log 1}{(1/e)-1} = -c$ or $\dfrac{1-0}{(1-e)/e} = -c$ or $\dfrac{e}{1-e} = -c$

ie., $c = \dfrac{e}{e-1} \approx \dfrac{2.7}{1.7} \approx 1.6$

$c = 1 \cdot 6 \in (1, e)$ since $e \approx 2.7$. Thus Cauchy's mean value theorem is verified.

>> We have Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Let $f(x) = \sin x$; $g(x) = \cos x$

$f'(x) = \cos x$; $g'(x) = -\sin x$

$f(x)$ and $g(x)$ are continuous in $[a, b]$, differentiable in (a, b)

$\cdot g'(x) \neq 0 \ \forall \ x \in (a, b)$ since $0 < a < b$

Hence the theorem becomes,

$$\frac{\sin b - \sin a}{\cos b - \cos a} = \frac{\cos c}{-\sin c}$$

ie, $-\sin b \sin c + \sin c \sin a = \cos b \cos c - \cos c \cos a$

or $\cos c \cos a + \sin c \sin a = \cos b \cos c + \sin b \sin c$

ie, $\cos(c-a) = \cos(b-c) \Rightarrow c - a = b - c$ or $2c = a + b$

. **$c = (a+b)/2$ is the arithmetic mean of a and b. $c \in (a, b)$**

Note *In order to find c we can also use the transformation formulae The simplification is as follows*

From (1) $\quad -\cot c = \dfrac{2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{b+a}{2}\right)}{-2\sin\left(\frac{b-a}{2}\right)\sin\left(\frac{b+a}{2}\right)} \quad \cot\left(\frac{b+a}{2}\right)$

Thus $c = (b+a)/2$

>> We have Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

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Let $\quad f(x) = 1/x^2 \quad ; \; g(x) = 1/x$

$$f'(x) = -2/x^3, \quad g'(x) = -1/x^2$$

$f(x)$ and $g(x)$ are continuous in $[a, b]$, differentiable in (a, b) and $g'(x) \neq 0$ $\forall \; x \in (a, b)$.

Hence the theorem becomes,

$$\frac{1/b^2 - 1/a^2}{1/b - 1/a} = \frac{-2/c^3}{-1/c^2} \quad \text{or} \quad \frac{a^2 - b^2/a^2 b^2}{a - b/ab} = \frac{2}{c}$$

ie., $$\frac{(a-b)(a+b)ab}{(a-b)\,a^2 b^2} = \frac{2}{c} \quad \text{or} \quad \frac{a+b}{ab} = \frac{2}{c}$$

Thus $\quad c = \dfrac{2ab}{a+b}$ **is the harmonic mean between a and b. $c \in (a, b)$**

Suppose we need to establish an inequality of the form $f_1(x) > f_2(x) > f_3(x)$ where $x > 0$, we take $F(x) = f_1(x) - f_2(x)$, $G(x) = f_2(x) - f_3(x)$ and apply Lagrange's mean value theorem for $F(x)$ and also for $G(x)$ in the interval $[0, x]$

>> **Case-(i)** Let $f(x) = x - \log(1+x)$

$\therefore \quad f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x}$; since $x > 0$, $f'(x) > 0$

$f(x)$ is continuous in $[0, x]$ differentiable in $(0, x)$

Applying Lagrange's mean value theorem for this $f(x)$ we have

$$f(x) = f(0) + (x - 0) f'(c) \quad \text{But } f(0) = 0$$

$$f(x) = x f'(c). \quad \text{Also } f'(c) > 0 \text{ since } c \in (0, x)$$

Hence $f(x) > 0$. That is $x - \log(1+x) > 0$

$$\therefore \quad x > \log(1+x) \qquad \ldots (1)$$

Case-(ii) Let $f(x) = \log(1+x) - x + (x^2/2)$

$$f'(x) = \frac{1}{1+x} - 1 + x = \frac{1 - 1 - x + x + x^2}{1+x} = \frac{x^2}{1+x}$$

Clearly $f'(x) > 0$. Again applying the theorem in $[0, x]$ we have $f(x) = f(0) + x f'(c)$ But $f(0) = 0$

Also $f'(x) > 0 \quad \Rightarrow \quad f'(c) > 0$ since $c \in (0, x)$

Hence we have again $f(x) > 0$

ie., $\quad \log(1+x) - x + (x^2/2) > 0$

$$\therefore \quad \log(1+x) > x - (x^2/2) \qquad \ldots (2)$$

Thus by combining (1) and (2) we have

$$\mathbf{x > \log(1+x) > x - (x^2/2)}$$

>> We shall establish the equivalent form of the desired result,

$$x > \log(1+x) > \frac{x}{1+x}$$

$x > \log(1+x)$ is same as Case-(i) of the previous problem.

Now, let $f(x) = \log(1+x) - \dfrac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \left\{ \frac{(1+x)\cdot 1 - x \cdot 1}{(1+x)^2} \right\} = \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

$$f'(x) = \frac{(1+x)-1}{(1+x)^2} = \frac{x}{(1+x)^2} \quad \text{Clearly } f'(x)>0 \text{ since } x>0.$$

Also $f(x)$ is continuous in $[0, x]$ and differentiable in $(0, x)$.

Applying Lagrange's mean value theorem for this $f(x)$ in $[0, x]$ we have,

$$f(x) = f(0)+(x-0)f'(c) \quad \text{But } f(0) = 0$$

$$f(x) = xf'(c); \ f'(x)>0 \Rightarrow f'(c) > 0 \text{ and hence } f(x)>0$$

ie., $$\log(1+x)-\frac{x}{1+x}>0 \text{ or } \log(1+x)>\frac{x}{1+x}$$

Since we also have $x>\log(1+x)$, combining these we get

$$\mathbf{x > \log(1+x) > \frac{x}{1+x} \quad \text{or} \quad \frac{x}{1+x} < \log(1+x) < x}$$

>> Let $f(x) = \cos x-1+(x^2/2)$

$$f'(x) = -\sin x+x$$

Since $x>0$, $\sin x<x$ or $x-\sin x>0$ $\therefore$ $f'(x)>0$

The function $f(x)$ is continuous in $[0, x]$, differentiable in $(0, x)$

Hence by applying Lagrange's mean value theorem for $f(x)$ in $[0, x]$ we have,

$$f(x) = f(0)+(x-0)f'(c)$$

But $f(0) = 0 \ \therefore \ f(x) = xf'(c).\ f'(x)>0 \Rightarrow f'(c)>0$

Hence $f(x)>0$. That is, $\cos x-1+(x^2/2)>0$

Thus $\mathbf{\cos x > 1-(x^2/2)}$

>> Let $f(x) = \sin^{-1}x$ $\therefore$ $f'(x) = \frac{1}{\sqrt{1-x^2}}$

$f(x)$ is continuous in $[a, b]$ and differentiable in (a, b).

Applying Lagrange's mean value theorem for $f(x)$ in $[a, b]$ we get when $a<c<b$

$$\frac{\sin^{-1}b-\sin^{-1}a}{b-a} = \frac{1}{\sqrt{1-c^2}} \quad (1)$$

Now $a<c \Rightarrow a^2<c^2 \Rightarrow -a^2>-c^2 \Rightarrow 1-a^2>1-c^2$

Hence $\dfrac{1}{\sqrt{1-a^2}}<\dfrac{1}{\sqrt{1-c^2}}$ ... (2)

Also $c<b$. On similar lines, $\dfrac{1}{\sqrt{1-c^2}}<\dfrac{1}{\sqrt{1-b^2}}$ ... (3)

Combining (2) and (3) we get,

$$\frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}}$$

or $$\frac{1}{\sqrt{1-a^2}}<\frac{\sin^{-1}b-\sin^{-1}a}{b-a}<\frac{1}{\sqrt{1-b^2}}$$ by using (1).

On multiplying by $(b-a)$ which is positive, we have

$$\frac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{b-a}{\sqrt{1-b^2}}$$

(*We need to take* $f(x)=\tan^{-1}x$)

>> Let $f(x)=\sin x$. By applying Lagrange's mean value theorem for $f(x)$ in $(x,\ x+h)$ we get,

$$\frac{\sin(x+h)-\sin x}{x+h-x}=f'(c) \quad ; \quad \text{But } f'(x)=\cos x$$

$$\therefore \quad \frac{\sin(x+h)-\sin x}{h}=\cos c$$

Thus $\mathbf{\sin(x+h)-\sin x=h\cos c}$ as required

>> Let $f(x)=\sin x$ and $g(x)=\cos x$

$$f'(x)=\cos x \text{ and } g'(x)=-\sin x$$

Applying Cauchy's mean value theorem we have,

$$\frac{\sin b - \sin a}{\cos b - \cos a} = \frac{\cos c}{-\sin c} \quad \text{or} \quad \frac{\sin b - \sin a}{\cos b - \cos a} = -\cot c$$

Thus $\quad \dfrac{\sin b - \sin a}{\cos a - \cos b} = \cot c$ as required.

Verify Rolle's theorem for the following functions.

1. $f(x) = \sin x / e^x$ in $[0, \pi]$
2. $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$
3. $f(x) = (x-a)[(x-a)(x-b)]^2$ in $[a, b]$

Verify Lagrange's mean value theorem for the following functions.

4. $f(x) = \log x$ in $[e, e^2]$
5. $f(x) = (x-2)(x+2)(x-3)$ in $[1, 4]$
6. $f(x) = px^2 + qx + r$ in $[a, b]$

Verify Cauchy's mean value theorem for the following pairs of functions.

7. $\sin x$ and $\cos x$ in $[\pi/4, 3\pi/4]$
8. e^x and $1/e^x$ in $[a, b]$
9. $f(x)$ and $2f'(x)$ in $[a, b]$, where $f(x) = \sqrt{x}$

10. Show that if $0 < a < b$,

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

and hence deduce that

$$\frac{5\pi + 4}{20} < (\tan^{-1} 2) < \frac{\pi + 2}{4}$$

1. $c = \pi/4$	**2.** $c = -2$	**3.** $c = (2a + 3b)/5$
4. $c = e^2 - e$	**5.** $c = 1 + \sqrt{3}$	**6.** $c = (a + b)/2$
7. $c = \pi$	**8.** $c = (a + b)/2$	**9.** $c = \sqrt{ab}$

We have Taylor's theorem in the form

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$

Evidently the expression in the RHS contains $(n+1)$ terms and we denote $R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h)$ which is called the remainder after n terms

Let $a+h = x$ or $h = (x-a)$. If x is close enough to 'a' then h will be very small and $R_n \to 0$ as $n \to \infty$. As $n \to \infty$ the number of terms increase indefinitely and we have an infinite series expansion of $f(x)$ in powers of $(x-a)$ given by

$$\boxed{f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots}$$

This is called ***Taylor's series expansion of*** $f(x)$ about the point 'a'

In particular if $a=0$ we have,

$$\boxed{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots}$$

This is called ***Maclaurin's series expansion*** of $f(x)$

For convenience we shall use the notation

$y(x)$ for $f(x)$ and $y_1(x)$, $y_2(x)$, $y_3(x)$ ... respectively for $f'(x)$, $f''(x)$, $f'''(x)$, ... so that we have

$$y(x)=y(a)+(x-a)\,y_1(a)+\frac{(x-a)^2}{2!}\,y_2(a)+\ldots \qquad \text{[\textit{Taylor's expansion}]}$$

$$y(x)=y(0)+x\,y_1(0)+\frac{x^2}{2!}\,y_2(0)+\ldots \qquad \text{[\textit{Maclaurin's expansion}]}$$

➲ We need to find successive derivatives of the given $y(x)$ and evaluate them at the given point $x=a$ for obtaining the Taylor's expansion and evaluate at $x=0$ for obtaining the Maclaurin's expansion

➲ To reduce the computational work we must prefer to use indirect methods for obtaining various derivatives of the given function which we are familiar in the discussion of the topic ***'Successive differentiation'***

>> We have Taylor's expansion about $x = a$ given by

$$y(x) = y(a) + (x-a)\,y_1(a) + \frac{(x-a)^2}{2!}\,y_2(a) + \ldots$$

By data, $y(x) = \log_e x$; $a = 1$

$y(1) = \log_e 1 = 0$ Differentiating $y(x)$ successively we get,

$$y_1(x) = \frac{1}{x} \quad \therefore\ y_1(1) = 1 \quad ; \quad y_3(x) = \frac{2}{x^3} \quad \therefore\ y_3(1) = 2$$

$$y_2(x) = -\frac{1}{x^2} \quad \therefore\ y_2(1) = -1 \quad ; \quad y_4(x) = -\frac{6}{x^4} \quad \therefore\ y_4(1) = -6$$

Taylor's series upto fourth degree term with $a = 1$ is given by

$$y(x) = y(1) + (x-1)\,y_1(1) + \frac{(x-1)^2}{2!}\,y_2(1) + \frac{(x-1)^3}{3!}\,y_3(1) + \frac{(x-1)^4}{4!}\,y_4(1)$$

Hence, $\log_e x = 0 + (x-1)\,1 + \dfrac{(x-1)^2}{2}(-1) + \dfrac{(x-1)^3}{6}(2) + \dfrac{(x-1)^4}{24}(-6)$

Thus $\quad \boldsymbol{\log_e x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dfrac{(x-1)^4}{4}}$

Now putting $x = 1.1$ we have

$$\log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = \mathbf{0.0953}$$

>> Taylor's expansion in powers of $(x-1)$ is given by

$$y(x) = y(1) + (x-1)\,y_1(1) + \frac{(x-1)^2}{2!}\,y_2(1) + \frac{(x-1)^3}{3!}(1) + \frac{(x-1)^4}{4!}\,y_4(1) + \ldots$$

$$y = \tan^{-1} x \qquad \therefore \quad y(1) = \tan^{-1} 1 = \pi/4$$

$$y_1 = \frac{1}{1+x^2} \qquad \therefore \quad y_1(1) = \frac{1}{2}\,,$$ (*We donot prefer direct differentiation*)

We have $\qquad (1+x^2)\,y_1 = 1 \qquad \ldots(1)$

We have to successively differentiate to obtain expressions involving y_2, y_3, y_4 and evaluate them at $x = 1$

Hence we have on differentiating (1),

$$(1+x^2)\,y_2 + 2\,x y_1 = 0 \qquad \ldots(2)$$

Putting $x = 1$; $\; 2y_2(1) + 2\cdot 1\cdot \frac{1}{2} = 0 \qquad \therefore \quad y_2(1) = -1/2$

Differentiating (2) w.r.t x, we get,

$$(1+x^2)\,y_3 + 4\,x y_2 + 2y_1 = 0 \qquad \ldots(3)$$

Putting $x = 1$; $\; 2y_3(1) - 2 + 1 = 0 \qquad \therefore \quad y_3(1) = 1/2$

Differentiating (3) w.r.t x, we get,

$$(1+x^2)\,y_4 + 6\,x y_3 + 6y_2 = 0 \qquad \ldots(4)$$

Putting $x = 1$; $\; 2y_4(1) + 3 - 3 = 0 \qquad \therefore \quad y_4(1) = 0$

Substituting these values in the expansion we get

$$\tan^{-1} x = \frac{\pi}{4} + \frac{1}{2}\left\{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6}\right\}$$

[illegible]

We have by Taylor's theorem,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots$$

Taking $a = \pi/4$, $f(\pi/4 + h) = \sin(\pi/4 + h) \Rightarrow f(x) = \sin x$

or $\qquad y(x) = \sin x$

$$y(\pi/4 + h) = y(\pi/4) + h\, y_1(\pi/4) + \frac{h^2}{2!} y_2(\pi/4) + \frac{h^3}{3!} y_3(\pi/4) + \frac{h^4}{4!} y_4(\pi/4) \qquad \ldots(1)$$

Consider $\; y(x) = \sin x \qquad \therefore \qquad y(\pi/4) = 1/\sqrt{2}$

$y_1(x) = \cos x \qquad \therefore \qquad y_1(\pi/4) = 1/\sqrt{2}$

$y_2(x) = -\sin x \qquad \therefore \qquad y_2(\pi/4) = -1/\sqrt{2}$

$y_3(x) = -\cos x \qquad \therefore \qquad y_3(\pi/4) = -1/\sqrt{2}$

$$y_4(x) = \sin x \qquad \therefore \qquad y_4(\pi/4) = 1/\sqrt{2}$$

Substituting these values in (1) we obtain

$$\sin(\pi/4 + h) = \frac{1}{\sqrt{2}}\left\{1 + h - \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!}\right\} \qquad \ldots (2)$$

To find $\sin 50^\circ$ we have to take $h = 5^\circ$

i.e., $h = 5(\pi/180)$ radians $= \pi/36 = 0.087$

Substituting $h = 0.087$ in the R.H.S of (2) we obtain,

$$\sin 50^\circ = 0.7659$$

58. *Obtain Taylor's series expansion of $\log(\cos x)$ about the point $x = \pi/3$ upto the 4th degree term.*

>> Taylor's expansion of $y(x)$ about $x = \pi/3$ is given by

$$y(x) = y(\pi/3) + (x - \pi/3)\, y_1(\pi/3) + \frac{(x-\pi/3)^2}{2!}\, y_2(\pi/3)$$

$$+ \frac{(x-\pi/3)^3}{3!}\, y_3(\pi/3) + \frac{(x-\pi/3)^4}{4!}\, y_4(\pi/3) + \cdots \qquad \ldots (1)$$

Let $\quad y(x) = \log(\cos x)$

$$y(\pi/3) = \log[\cos(\pi/3)] = \log(1/2) = -\log 2$$

$$y_1 = \frac{1}{\cos x} \cdot -\sin x$$

i.e., $\quad y_1 = -\tan x \qquad \therefore \quad y_1(\pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$$y_2 = -\sec^2 x = -(1 + \tan^2 x)$$

i.e., $\quad y_2 = -(1 + y_1^2) \qquad \therefore \quad y_2(\pi/3) = -[1 + (\sqrt{3})^2] = -4$

$$y_3 = -2\, y_1 y_2 \qquad \therefore \quad y_3(\pi/3) = -2 \cdot -\sqrt{3} \cdot -4 = -8\sqrt{3}$$

$$y_4 = -2[y_1 y_3 + y_2^2] \qquad \therefore \quad y_4(\pi/3) = -2[-\sqrt{3} \cdot -8\sqrt{3} + 16] = -80$$

Substituting these values in (1) we have,

$$\log(\cos x) = -\log 2 - (x - \pi/3)\sqrt{3} - \frac{(x-\pi/3)^2}{2} \cdot 4$$

$$- \frac{(x-\pi/3)^3}{6} \cdot 8\sqrt{3} - \frac{(x-\pi/3)^4}{24} \cdot 80 + \cdots$$

Thus $\log(\cos x) = -\log 2 - \sqrt{3}\,(x-\pi/3) - 2\,(x-\pi/3)^2$

$$- \frac{4}{\sqrt{3}}(x-\pi/3)^3 - \frac{10}{3}(x-\pi/3)^4$$

>> We have $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots$

$$y = \sin^{-1} x \qquad \therefore \quad y(0) = \sin^{-1} 0 = 0$$

$$y_1 = \frac{1}{\sqrt{1-x^2}} \qquad \ldots (1)$$

[*Finding y_2, y_3, y_4, etc. becomes very difficult through direct differentiation. Hence we shall adopt the technique of avoiding denominator by cross multiplying and squaring for avoiding square root*]. We have $y_1(0) = 1$.

Thus we have $(1-x^2)\,y_1^2 = 1$

Differentiating now w.r.t. x, we get,

$(1-x^2)\,2\,y_1\,y_2 + y_1^2(-2x) = 0$. Dividing by $2y_1$ we get,

$$(1-x^2)\,y_2 - xy_1 = 0 \qquad \ldots (2)$$

Now putting $x = 0$ we have $1 \cdot y_2(0) - 0 = 0 \qquad \therefore \quad y_2(0) = 0$

[*Here* y_1, y_2 *is to be understood as* $y_1(x)$, $y_2(x)$].

Now differentiating (2) w.r.t. x we obtain

$$(1-x^2)\,y_3 + y_2(-2x) - [\,xy_2 + y_1 \cdot 1\,] = 0$$

ie, $$(1-x^2)\,y_3 - 3xy_2 - y_1 = 0 \qquad \ldots (3)$$

Putting $x = 0$ we have $y_3(0) - 0 - 1 = 0, \qquad \therefore \quad y_3(0) = 1$

Differentiating (3) again w.r.t. x we obtain,

$$(1-x^2)\,y_4 + y_3(-2x) - 3\,[\,xy_3 + y_2 \cdot 1\,] - y_2 = 0$$

ie., $$(1-x^2)\,y_4 - 5xy_3 - 4y_2 = 0 \qquad \ldots (4)$$

Putting $x = 0$ we have $y_4(0) - 0 - 0 = 0 \qquad \therefore \quad y_4(0) = 0$

Differentiating (4) again w.r.t. x we have,

$$(1-x^2)\,y_5 + y_4(-2x) - 5[\,xy_4 + y_3 \,.\, 1\,] - 4y_3 = 0$$

i.e., $$(1-x^2)\,y_5 - 7xy_4 - 9y_3 = 0 \tag{5}$$

Putting $x = 0$ we have $y_5(0) - 0 - 9\,.\,1 = 0 \quad \therefore \quad y_5(0) = 9$

Substituting these values in the expansion of $y(x)$ we have,

$$\sin^{-1} x = 0 + x \cdot 1 + \frac{x^2}{2} \cdot 0 + \frac{x^3}{6} \cdot 1 + \frac{x^4}{24} \cdot 0 + \frac{x^5}{120} \cdot 9$$

Thus $\quad \mathbf{\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40}}$

Note : [illegible]

[illegible] ascending powers of x upto the first three non vanishing terms

>> We have Maclaurin's expansion,

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + \frac{x^4}{4!}\,y_4(0) + \ldots$$

Let $\quad y = e^{\sin x} \qquad \therefore \quad y(0) = e^0 = 1$

$y_1 = e^{\sin x} \cos x$ or $y_1 = y \cos x \quad \therefore \quad y_1(0) = y(0) \cdot \cos 0 = 1$

$y_2 = -y \sin x + \cos x \cdot y_1 \qquad \therefore \quad y_2(0) = 0 + 1 = 1$

$y_3 = -(\,y \cos x + y_1 \sin x\,) + (\,\cos x\; y_2 - y_1 \sin x\,)$

$\quad = -y_1 - 2y_1 \sin x + \cos x \cdot y_2 \qquad \therefore \quad y_3(0) = -1 - 0 + 1 = 0$

$y_4 = -y_2 - 2(\,y_1 \cos x + \sin x\;\, y_2\,) + (\,\cos x\; y_3 - \sin x\;\, y_2\,)$

$\quad = -y_2 - 2\,y_1 \cos x - 3 \sin x\; y_2 + \cos x \cdot y_3$

$y_4(0) = -1 - 2 - 0 + 0 = -3 \qquad \therefore \quad y_4(0) = -3$

Hence by substituting these values in the expansion of $y(x)$ we get,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} \quad \frac{x^4}{8}$$

61. *Expand $\log(1+\sin x)$ in powers of x by Maclaurin's theorem upto the term containing x^4*

>> We have Maclaurin's expansion

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}\ldots$$

Consider $y = \log(1+\sin x)$ $\qquad \therefore \quad y(0) = \log 1 = 0$

$$y_1 = \frac{\cos x}{1+\sin x} \qquad \therefore \quad y_1(0) = 1$$

ie., $(1+\sin x)\,y_1 = \cos x \qquad \ldots(1)$

Differentiating w.r.t x we get,

$$(1+\sin x)\,y_2 + \cos x\;y_1 = -\sin x \qquad \ldots(2)$$

At $x=0$ we get $y_2(0)+1 = 0$ $\qquad \therefore \quad y_2(0) = -1$

Differentiating (2) again we get,

$$(1+\sin x)\,y_3 + 2y_2\cos x - y_1\sin x = -\cos x \qquad \ldots(3)$$

At $x = 0$ we get, $y_3(0) - 2 - 0 = -1$ $\qquad \therefore \quad y_3(0) = 1$.

Differentiating (3) again we get.

$$(1+\sin x)\,y_4 + \cos x\;y_3 + 2(-y_2\sin x + \cos x\;y_3) - (y_1\cos x + \sin x\;y_2) = \sin x$$

ie., $(1+\sin x)\,y_4 + 3\cos x\,y_3 - 3y_2\sin x - y_1\cos x = \sin x$

At $x = 0$ we get, $y_4(0) + 3 - 0 - 1 = 0$ $\qquad \therefore \quad y_4(0) = -2$

Thus by substituting these values in the expansion we get

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$$

term containing x^5 using Maclaurin's series

Expand $\log(\sec x)$ in ascending powers of x upto the first three non vanishing terms.

>> $$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots$$

$y = \log(\sec x) \quad \therefore \quad y(0) = \log 1 = 0$

$$y_1 = \frac{\sec x \tan x}{\sec x} \quad i.e., \; y_1 = \tan x \quad \therefore \; y_1(0) = 0$$

$$y_2 = \sec^2 x \quad \therefore \; y_2(0) = 1.$$

Now $\quad y_2 = 1 + \tan^2 x = 1 + y_1^2$

Differentiating this w.r.t. x successively we have,

$$y_3 = 2y_1 y_2 \qquad \therefore \; y_3(0) = 0$$

$$y_4 = 2(y_1 y_3 + y_2^2) \qquad \therefore \; y_4(0) = 2$$

$$y_5 = 2(y_1 y_4 + y_2 y_3 + 2y_2 y_3) = 2y_1 y_4 + 6y_2 y_3 \qquad \therefore \; y_5(0) = 0$$

$$y_6 = 2(y_1 y_5 + y_2 y_4) + 6(y_2 y_4 + y_3^2)$$

i.e. $\quad y_6 = 2y_1 y_5 + 8y_2 y_4 + 6y_3^2 \qquad \therefore \; y_6(0) = 16$

Substituting these values in the expansion of $y(x)$ we get,

$$\log(\sec x) = \text{[illegible]}$$

Thus $\quad \log(\sec x) = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45}$

63. Expand $\tan^{-1}(1+x)$ as far as the term containing x^3

>> $\quad y(x) = y(0) + x\,y_1(0) + \dfrac{x^2}{2!}\,y_2(0) + \dfrac{x^3}{3!}\,y_3(0) + \cdots$

Consider $\quad y = \tan^{-1}(1+x) \qquad \therefore \; y(0) = \tan^{-1}(1) = \pi/4$

$$y_1 = \frac{1}{1+(1+x)^2} = \frac{1}{x^2+2x+2} \qquad \therefore \; y_1(0) = 1/2$$

i.e., $\quad (x^2+2x+2)\,y_1 = 1 \qquad \ldots (1)$

Differentiating w.r.t. x,

$$(x^2+2x+2)\,y_2 + 2(x+1)\,y_1 = 0 \qquad \ldots (2)$$

At $\quad x = 0$ we get $2y_2(0) + 2\cdot(1/2) = 0 \qquad \therefore \; y_2(0) = -1/2$

Differentiating (2) w.r.t. x we have,

$$(x^2+2x+2)\,y_3 + 2(x+1)\,y_2 + 2(x+1)\,y_2 + 2y_1 = 0$$

ie., $\quad (x^2+2x+2)\,y_3+4\,(x+1)\,y_2+2y_1 = 0 \qquad \ldots (3)$

At $x=0$ we get, $2y_3(0)+4\cdot(-1/2)+2\cdot(1/2)=0 \quad \therefore \quad y_3(0)=1/2$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\tan^{-1}(1+x) = \frac{\pi}{4}+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{12}$$

>> We have, $y(x) = y(0)+x\,y_1(0)+\frac{x^2}{2!}\,y_2(0)+\ldots$

Let $\quad y = \sqrt{1+\sin 2x} = \sqrt{\cos^2 x+\sin^2 x+2\sin x\cos x}$

$= \sqrt{(\cos x+\sin x)^2} = \cos x+\sin x$

Thus $\quad y = \cos x+\sin x \qquad \therefore \quad y(0)=1$

$y_1 = -\sin x+\cos x \qquad \therefore \quad y_1(0)=1$

$y_2 = -\cos x-\sin x = -y \quad$ ie., $y_2=-y \qquad \therefore \quad y_2(0)=-1$

$y_3 = -y_1 \;;\; y_3(0)=-1. \quad y_4=-y_2 \qquad \therefore \quad y_4(0)=1$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\sqrt{1+\sin 2x} = 1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}$$

Remark *We have used a technique to simplify the given function leading to a very simple form thereby the problem is completed easily. The usual procedure of squaring $y(x)$ and differentiating thereon can also be done.*

>> We have $y(x) = y(0)+x\,y_1(0)+\frac{x^2}{2!}\,y_2(0)+\frac{x^3}{3!}\,y_3(0)+\ldots$

Let $\quad y=\log(1+x) \quad \therefore \quad y(0)=\log 1=0$

$y_1 = \frac{1}{1+x} \qquad \therefore \quad y_1(0)=1 \;;\quad y_2=\frac{-1}{(1+x)^2} \qquad \therefore \quad y_2(0)=-1$

$y_3 = \frac{2}{(1+x)^3} \qquad \therefore \quad y_3(0)=2 \;;\quad y_4=\frac{-6}{(1+x)^4} \qquad \therefore \quad y_4(0)=-6$

$$y_5 = \frac{24}{(1+x)^5} \qquad \therefore \quad y_5(0) = 24$$

Substituting these values in the expansion we get,

$$\log(1+x) = 0 + x \cdot 1 + \frac{x^2}{2}(-1) + \frac{x^3}{6} \cdot 2 + \frac{x^4}{24}(-6) + \frac{x^5}{120} \cdot 24 - \cdots$$

Thus $\quad \mathbf{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots}$ $\quad \ldots(1)$

Next, $\quad \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$ $\quad \ldots(2)$

Changing x to $-x$ in (1) we obtain

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots \qquad \ldots(3)$$

Using (1) and (3) in (2) we have

$$\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left\{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right)\right\}$$

i.e., $\quad \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left(2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

Thus $\quad \mathbf{\log\sqrt{1+x/1-x} = x + (x^3/3) + (x^5/5) + \cdots}$

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$$

Let $\quad y = \tan(\pi/4 + x) \qquad \therefore \quad y(0) = \tan(\pi/4) = 1$

$y_1 = \sec^2(\pi/4 + x) = 1 + y^2 \qquad \therefore \quad y_1(0) = 2$

$y_2 = 2yy_1 \qquad \therefore \quad y_2(0) = 4$

$y_3 = 2(yy_2 + y_1^2) \qquad \therefore \quad y_3(0) = 2(4+4) = 16$

$y_4 = 2(y\,y_3 + 3\,y_1\,y_2) \qquad \therefore \quad y_4(0) = 2(16+24) = 80$

Substituting these values in the expansion of $y(x)$ we have,

$$\tan(\pi/4 + x) = 1 + x \cdot 2 + \frac{x^2}{2} \cdot 4 + \frac{x^3}{6} \cdot 16 + \frac{x^4}{24} \cdot 80$$

Thus $\quad \mathbf{\tan(\pi/4 + x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4}$

>> $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + ..$

Let $y = \log\tan(\pi/4 + x)$; $y(0) = \log 1 = 0$

Also $e^y = \tan(\pi/4 + x)$

Differentiating w.r.t. x we get,

$$e^y\,y_1 = \sec^2(\pi/4 + x) = 1 + (e^y)^2$$

ie., $e^y\,y_1 = 1 + e^{2y}$ ... (1)

At $x = 0,\ 1 \cdot y_1(0) = 2$ $\therefore\ y_1(0) = 2$

Differentiating (1) w.r.t. x we get,

$$e^y\,y_2 + e^y\,y_1^2 = 2\,e^{2y}\,y_1 \quad \text{or} \quad y_2 + y_1^2 = 2\,e^y\,y_1 \qquad \dots (2)$$

At $x = 0$, $y_2(0) + 4 = 4$ $\therefore\ y_2(0) = 0$

Differentiating (2) w.r.t. x we get,

$$y_3 + 2\,y_1\,y_2 = 2\,(e^y\,y_2 + e^y\,y_1^2) \qquad \dots (3)$$

At $x = 0$, $y_3(0) + 0 = 2(0 + 4)$ $\therefore\ y_3(0) = 8$

Differentiating (3) w.r.t. x we get,

$$y_4 + 2\,y_1\,y_3 + 2\,y_2^2 = 2\,(e^y\,y_3 + e^y\,y_1\,y_2 + 2\,e^y\,y_1\,y_2 + e^y\,y_1^3)$$

ie., $y_4 + 2\,y_1\,y_3 + 2\,y_2^2 = 2\,e^y\,(y_3 + 3\,y_1\,y_2 + y_1^3)$ ... (4)

At $x = 0$, $y_4(0) + 2 \times 2 \times 8 + 0 = 2(8 + 0 + 8)$ $\therefore\ y_4(0) = 0$

Differentiating (4) w.r.t. x we get,

$$y_5 + 2\,y_1\,y_4 + 2\,y_2\,y_3 + 4\,y_2\,y_3 = 2\,e^y\,(y_4 + 3\,y_1\,y_3 + 3\,y_2^2 + 3\,y_1^2\,y_2) + 2\,e^y\,y_1\,(y_3 + 3\,y_1\,y_2 + y_1^3)$$

At $x = 0$, $y_5(0) = 2(48) + 2 \cdot 2(8 + 8) = 160$

Substituting these values in the expansion of $y(x)$ we have,

$$\log\tan(\pi/4 + x) = x \cdot 2 + \frac{x^3}{6} \cdot 8 + \frac{x^5}{120} \cdot 160 + \dots$$

Thus $\mathbf{\log\tan(\pi/4 + x) = 2x + \frac{4x^3}{3} + \frac{4x^5}{3} + \dots}$

08 [illegible]

$$>> \quad y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots$$

Let $\quad y = e^{x \sin x} \qquad\qquad y(0) = 1$

$\qquad y_1 = e^{x \sin x}(x \cos x + \sin x) \qquad\qquad y_1(0) = 0$

ie, $\quad y_1 = y(x \cos x + \sin x)$

Now $\quad y_2 = y(-x \sin x + 2\cos x) + y_1(x \cos x + \sin x)$

Hence $\quad y_2(0) = 1(0+2) + 0 \qquad\qquad y_2(0) = 2$

Next, $\quad y_3 = y(-x \cos x - 3\sin x) + y_1(-x \sin x + 2\cos x)$

$$+\; y_1(-x \sin x + 2\cos x) + y_2(x \cos x + \sin x)$$

ie, $\quad y_3 = y(-x \cos x - 3\sin x) + 2y_1(-x \sin x + 2\cos x) + y_2(x \cos x + \sin x)$

Hence $\quad y_3(0) = 0 + 0 + 0 \qquad\qquad y_3(0) = 0$

Next $\quad y_4 = y(x \sin x - 4\cos x) + y_1(-x \cos x - 3\sin x)$

$$+\; 2y_1(-x \cos x - 3\sin x) + 2y_2(-x \sin x + 2\cos x)$$

$$+\; y_2(-x \sin x + 2\cos x) + y_3(x \cos x + \sin x)$$

Hence $\quad y_4(0) = -4 + 0 + 0 + 8 + 4 = 8 \qquad\qquad y_4(0) = 8$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$e^{x \sin x} = 1 + x^2 + (x^4/3) + \ldots$$

[illegible] $\log(1+e^x)$

$$>> \quad y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots$$

Let $\quad y = \log(1+e^x) \qquad \therefore \quad y(0) = \log_e 2$

$$y_1 = \frac{e^x}{1+e^x} \qquad \therefore \quad y_1(0) = \frac{1}{2}$$

ie, $\quad (1+e^x)\,y_1 = e^x \qquad\qquad \ldots (1)$

Differentiating w.r.t. x we get,

$$(1+e^x)\,y_2+e^x\,y_1 = e^x \qquad \ldots(2)$$

At $\quad x = 0\ ,\ 2\,y_2(0)+1/2 = 1 \qquad \therefore \quad y_2(0) = 1/4$

Differentiating (2) w.r.t. x we get,

$$(1+e^x)\,y_3+2\,e^x\,y_2+e^x\,y_1 = e^x \qquad \ldots(3)$$

At $\quad x = 0\ ,\ 2y_3(0)+1/2+1/2 = 1 \qquad \therefore \quad y_3(0) = 0$

Differentiating (3) w.r.t. x we get,

$$(1+e^x)\,y_4+3e^x\,y_3+3e^x\,y_2+e^x\,y_1 = e^x \qquad \ldots(4)$$

At $\quad x = 0\ ,\ 2y_4(0)+3/4+1/2 = 1 \qquad \therefore \quad y_4(0) = -1/8$

Thus by substituting these values in the expansion of $y(x)$ we get,

$$\mathbf{\log(1+e^x) = \log_e 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}}$$

>> Maclaurin's series is given by

$$y(x) = y(0)+x\,y_1(0)+\frac{x^2}{2!}\,y_2(0)+\frac{x^3}{3!}\,y_3(0)+\frac{x^4}{4!}\,y_4(0)+\cdots$$

Consider $y = y(x) = \log(1+\cos x)\ ;\ y(0) = \log_e 2$

$$y_1 = \frac{-\sin x}{1+\cos x} \qquad \therefore \quad y_1(0) = 0$$

ie., $\quad (1+\cos x)\,y_1 = -\sin x \qquad \ldots(1)$

Differentiating w.r.t x we have,

$$(1+\cos x)\,y_2-\sin x\ y_1 = -\cos x \qquad \ldots(2)$$

At $\quad x = 0,$ we get $2y_2(0)-0 = -1 \qquad \therefore \quad y_2(0) = -1/2$

Differentiating (2) again w.r.t x we have

$$(1+\cos x)\,y_3-\sin x\ y_2-[\sin x\ y_2+\cos x\ y_1] = \sin x$$

ie., $\quad (1+\cos x)\,y_3-2\sin x\ y_2-\cos x\ y_1 = \sin x \qquad \ldots(3)$

At $\quad x = 0,$ we get $2\,y_3(0)-0-0 = 0 \qquad \therefore \quad y_3(0) = 0$

Differentiating (3) again w.r.t x we have,

$$(1+\cos x)\,y_4 - \sin x\; y_3 - 2\,[\,\sin x\; y_3 + \cos x\; y_2\,] - [\,\cos x\; y_2 - \sin x\; y_1\,] = \cos x$$

ie., $\quad (1+\cos x)\,y_4 - 3\sin x\; y_3 - 3\cos x\; y_2 + \sin x\; y_1 = \cos x \qquad \ldots(4)$

At $\quad x = 0$, we get $2y_4(0) - 0 + 3/2 + 0 = 1 \quad \therefore \quad y_4(0) = -1/4$

Thus the required Maclaurin's series is given by

$$\mathbf{\log_e(1+\cos x) = \log_e 2 - \frac{x^2}{4} - \frac{x^4}{96} \cdots}$$

$$y = \log(1+\cos x) = \log[\,2\cos^2(x/2)\,]$$

ie., $\quad y = \log 2 + 2\log\cos(x/2) \qquad \therefore \quad y(0) = \log_e 2$

Now $\quad y_1 = -\tan(x/2) \qquad \therefore \quad y_1(0) = 0$

$$y_2 = -\frac{1}{2}\sec^2(x/2) \qquad \therefore \quad y_2(0) = -1/2$$

Also $\quad y_2 = -\frac{1}{2}[\,1+\tan^2(x/2)\,] = -\frac{1}{2}(1+y_1^2)$

$\therefore \quad y_3 = -\frac{1}{2}(2\,y_1 y_2) = -y_1 y_2 \qquad \therefore \quad y_3(0) = 0$

$$y_4 = -y_1 y_3 - y_2^2 \qquad \therefore \quad y_4(0) = -1/4$$

Thus $\quad \mathbf{\log(1+\cos x) = \log_e 2 - (x^2/4) - (x^4/96) \cdots}$

>> $\quad y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + ..$

Let $\quad y = a^x \qquad \therefore \quad y(0) = 1$

$$y_1 = a^x \log a = y\log a \qquad \therefore \quad y_1(0) = \log a$$

$$y_2 = y_1 \log a \qquad \therefore \quad y_2(0) = (\log a)^2$$

$$y_3 = y_2 \log a \qquad \therefore \quad y_3(0) = (\log a)^3 \quad \text{and so on}$$

Substituting these values in the expansion of $y(x)$ we get,

$$\mathbf{a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots}$$

72. Expand $\tan^{-1}$ [illegible]

[illegible]

$$>> \quad y(x) = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots$$

Let $\quad y = \tan^{-1} x \qquad \therefore \quad y(0) = \tan^{-1}(0) = 0$

$$y_1 = \frac{1}{1+x^2} \qquad \therefore \quad y_1(0) = 1$$

ie., $\quad (1+x^2)\, y_1 = 1 \qquad (1)$

Differentiating w.r.t. x we get,

$$(1+x^2)\, y_2 + 2xy_1 = 0 \qquad (2)$$

At $\quad x = 0$ we get, $\; y_2(0) = 0 \qquad \therefore \quad y_2(0) = 0$

Differentiating (2) again we get $\;(1+x^2)\, y_3 + 2x\, y_2 + 2x\, y_2 + 2y_1$ [illegible]

ie., $\quad (1+x^2)\, y_3 + 4xy_2 + 2y_1 = 0 \qquad (3)$

At $\quad x = 0$ we get, $\; y_3(0) = -2. \qquad \therefore \quad y_3(0) = -2$

Differentiating (3) again w.r.t.x, we get

$$(1+x^2)\, y_4 + 2xy_3 + 4xy_3 + 4y_2 + 2y_2 = 0$$

ie., $\quad (1+x^2)\, y_4 + 6xy_3 + 6y_2 = 0 \qquad (4)$

At $\quad x = 0$ we get, $\; y_4(0) = 0 \qquad \therefore \quad y_4(0) = 0$

Differentiating (4) again w.r.t. x we get,

$$(1+x^2)\, y_5 + 2xy_4 + 6xy_4 + 6y_3 + 6y_3 = 0$$

ie., $\quad (1+x^2)\, y_5 + 8xy_4 + 12y_3 = 0 \qquad \ldots (5)$

At $\quad x = 0$ we get, $\; y_5(0) = 24 \qquad \therefore \quad y_5(0) = 24$

Substituting these values in the expansion we have

$$\tan^{-1} x = x \cdot 1 + \frac{x^3}{6}(-2) + \frac{x^5}{120}(24)$$

Thus $\quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

Putting $\; x = 1 \;$ we get $\; \tan^{-1} 1 = 1 - 1/3 + 1/5 - 1/7 + \ldots$

ie., $\quad \pi/4 = 1 - 1/3 + 1/5 - \ldots \qquad$ Thus $\; \pi = 4(1 - 1/3 + 1/5 - \ldots)$

Note - 1 : *Expansion of functions*

(i) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ (ii) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ (iii) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

(iv) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

These functions by the substitution $x = \tan\theta$ *respectively become*

(i) $\pi/4 + \tan^{-1} x$ (ii) $2\tan^{-1} x$ (iii) $1/2 \cdot \tan^{-1} x$ (iv) $2\tan^{-1} x$.

Hence the work exhibited in the Problem - [illegible] has to be carried out for completing the problem in the case of functions (i) to (iv)

Note - 2 *Also in some cases we can venture to obtain a relation in* y, y_1, y_2 *and apply Leibnitz theorem to differentiate the result n times to obtain a relation in terms of* y_{n+2}, y_{n+1}, y_n. *Then by knowing* $y(0)$ *and* $y_1(0)$ *we can easily obtain* $y_2(0)$, $y_3(0)$, ... *by taking* $n = 0$, $n = 1, \ldots$

For example in the case of $y = \tan^{-1} x$ *we have the relation [Refer Problem - 13]*

$$(1+x^2)\, y_{n+2} + 2(n+1)\, x\, y_{n+1} + n(n+1)\, y_n = 0$$

At $x = 0$, $y_{n+2}(0) = -n(n+1)\, y_n(0)$ ... (1)

We have $y(0) = 0$, $y_1(0) = 1$ *where* $y = \tan^{-1} x$.

Putting $n = 0, 1, 2, 3 \ldots$ *in the relation (1) we get,* $y_2(0) = 0$,

$$y_3(0) = -2\, y_1(0) = -2 \ , \quad y_4(0) = -6\, y_2(0) = 0$$

$$y_5(0) = -12\, y_3(0) = (-12)(-2) = 24 \quad etc.$$

Note - 3 *We can adopt this method for Problem - 59, where* $y = \sin^{-1} x$

Let $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, $y(0) = 0$

$$\sqrt{1-x^2}\; y = \sin^{-1} x$$

Differentiating w.r.t. x we get,

$$\sqrt{1-x^2}\cdot y_1 + \frac{1}{2\sqrt{1-x^2}}(-2x)\, y = \frac{1}{\sqrt{1-x^2}}$$

or $\quad (1-x^2)\, y_1 - xy = 1 \quad ; \quad$ At $x = 0, \quad y_1(0) = 1$

Differentiating w.r.t. x again we get,

$$(1-x^2)\, y_2 - 2x\, y_1 - xy_1 - y = 0$$

or $\quad (1-x^2)\, y_2 - 3xy_1 - y = 0 \quad ; \quad$ At $x = 0, \quad y_2(0) = 0$

Applying Leibnitz theorem we have,

$$\left\{ (1-x^2)\, y_{n+2} + n\,(-2x)\, y_{n+1} + \frac{n(n-1)}{1 \cdot 2}(-2)\, y_n \right\} - 3\left\{ x\, y_{n+1} + n\, y_n \right\} - y_n = 0$$

ie., $\quad (1-x^2)\, y_{n+2} - (2n+3)\, x\, y_{n+1} - n^2 y_n + n\, y_n - 3ny_n - y_n = 0$

ie., $\quad (1-x^2)\, y_{n+2} - (2n+3)\, x\, y_{n+1} - (n+1)^2 y_n = 0$

At $x = 0$: $y_{n+2}(0) = (n+1)^2 y_n(0)$

$\therefore \quad y_3(0) = 4\, y_1(0) = 4 \,, \quad y_4(0) = 9\, y_2(0) = 0 \quad y_5(0) = 16\, y_3(0) = 64$ etc

We have the expansion,

$$y(x) = y(0) + x\, y_1(0) + \frac{x^2}{2!}\, y_2(0) + \ldots$$

ie., $$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x \cdot 1 + \frac{x^3}{6} \cdot 4 + \frac{x^5}{120} \cdot 64 + \ldots = x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \ldots$$

Thus $$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \ldots$$

Remark *We can as well apply Leibnitz theorem after finding a relationship involving y_1 so that we obtain a relationship involving y_{n+1}*

>> $\quad y(x) = y(0) + x\, y_1(0) + \dfrac{x^2}{2!}\, y_2(0) + \ldots$

Let $\quad y = e^{a \sin^{-1} x} \quad \therefore \quad y(0) = 1$

$$y_1 = e^{a \sin^{-1} x} \cdot \frac{a}{\sqrt{1-x^2}} \quad \text{or} \quad y_1 = \frac{ay}{\sqrt{1-x^2}} \quad \therefore \quad y_1(0) = a$$

Consider $\sqrt{1-x^2}\, y_1 = a\, y$ and by differentiating w.r.t. x we get,

$$\sqrt{1-x^2}\, y_2 + \frac{-2x}{2\sqrt{1-x^2}}\, y_1 = a\, y_1$$

ie., $\quad (1-x^2)\, y_2 - x\, y_1 = a\sqrt{1-x^2}\, y_1 = a\,(a\, y)$

ie., $\quad (1-x^2)\, y_2 - x\, y_1 = a^2\, y$

Applying Leibnitz theorem we have,

$$\left\{ (1-x^2)\, y_{n+2} + n\,(-2x)\, y_{n+1} + \frac{n\,(n-1)}{1\cdot 2}(-2)\, y_n \right\} - \{ x y_{n+1} + n \cdot 1 \cdot y_n \} = a^2\, y_n$$

ie., $\quad (1-x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2+a^2)\, y_n = 0$

At $x = 0$, $\; y_{n+2}(0) = (n^2+a^2)\, y_n(0)$

$$y_2(0) = a^2\, y(0) = a^2\; ; \quad y_3(0) = (1+a^2)\, y_1(0) = (1+a^2)\, a$$

$$y_4(0) = (4+a^2)\, y_2(0) = (4+a^2)\, a^2$$

$$y_5(0) = (9+a^2)\, y_3(0) = (9+a^2)\,(1+a^2)\, a \quad \text{etc.}$$

Thus by substituting these values in the expansion of $y(x)$ we have,

$$e^{a\sin^{-1}x} = 1 + ax + a^2\frac{x^2}{2!} + \frac{a(1+a^2)}{3!}x^3 + \frac{a^2(4+a^2)}{4!}x^4 + \frac{a(9+a^2)(1+a^2)}{5!}x^5 + \dots$$

>> $\quad y(x) = y(0) + x\, y_1(0) + \frac{x^2}{2!}\, y_2(0) + \dots$

Let $\quad y = e^{\tan^{-1}x} \qquad \therefore \quad y(0) = 1$

$$y_1 = \frac{e^{\tan^{-1}x}}{1+x^2} \quad ie.,\; y_1 = \frac{y}{1+x^2} \qquad \therefore \quad y_1(0) = 1$$

Consider $(1+x^2)\, y_1 = y$ and by applying Leibnitz theorem we have,

$$(1+x^2)\, y_{n+1} + n \cdot 2x \cdot y_n + \frac{n(n-1)}{1 \cdot 2} \cdot 2\, y_{n-1} = y_n$$

ie., $$(1+x^2)\, y_{n+1} + (2nx-1)\, y_n + n(n-1)\, y_{n-1} = 0$$

At $x = 0$, $y_{n+1}(0) = y_n(0) - n(n-1)\, y_{n-1}(0)$.

Hence we have,

$$y_2(0) = y_1(0) - 0 = 1$$

$$y_3(0) = y_2(0) - 2\, y_1(0) = 1 - 2 = -1$$

$$y_4(0) = y_3(0) - 6\, y_2(0) = -1 - 6 = -7$$

$$y_5(0) = y_4(0) - 12\, y_3(0) = -7 + 12 = 5$$

Substituting these values in the expansion of $y(x)$ we have,

$$e^{\tan^{-1} x} = 1 + x \cdot 1 + \frac{x^2}{2} \cdot 1 + \frac{x^3}{6} \cdot (-1) + \frac{x^4}{24}(-7) + \frac{x^5}{120} \cdot 5 \ldots$$

Thus $$e^{\tan^{-1} x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7}{24} x^4 + \frac{x^5}{24} \ldots$$

Note : *Maclaurin's expansions of the functions* $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, e^x *can be found easily and it is advisable to remember them. They are as follows.*

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ (ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iii) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$ (iv) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

(v) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

... the term containing x^4

>> $$y(x) = y(0) + x\, y_1(0) + \frac{x^2}{2!}\, y_2(0) + \ldots$$

We have $f(x) = y = x / \sin x$. Here $y(0)$ assumes $0/0$ form and we know that

$\lim\limits_{x \to 0} \frac{x}{\sin x} = 1 \quad \therefore \quad y(0) = 1$

y_1 also assumes $0/0$ form and we have to apply L' Hospital's rule to find $y_1(0)$. The process becomes highly tedious as we proceed. Hence we try to make use of the expansion of $\sin x$ and since $\sin x$ is in the denominator we cannot carry out any simplification. Hence we take the reciprocal of y and proceed as follows.

$$y = \frac{x}{\sin x} \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Now $\quad \dfrac{1}{y} = \dfrac{\sin x}{x} = \dfrac{1}{x}\left[x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots\right]$

ie., $\quad \dfrac{1}{y} = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \cdots$

Differentiating w.r.t. x we have,

$$\frac{-1}{y^2}y_1 = \frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots \qquad \ldots (1)$$

At $\quad x = 0, \quad -y_1(0) = 0 \quad$ since $\quad y(0) = 1 \quad \therefore \quad y_1(0) = 0$.

Differentiating (1) w.r.t. x we have,

$$\frac{-1}{y^2}y_2 + \frac{2}{y^3}y_1^2 = \frac{-2}{3!} + \frac{12x^2}{5!} - \frac{30x^4}{7!} + \cdots \qquad \ldots (2)$$

At $\quad x = 0, \; -y_2(0) + 0 = -1/3 \qquad \therefore \quad y_2(0) = 1/3$

Differentiating (2) w.r.t. x we have,

$$\frac{-1}{y^2}y_3 + \frac{2}{y^3}y_1y_2 + \frac{4}{y^3}y_1y_2 - \frac{6}{y^4}y_1^3 = \frac{24x}{5!} - \frac{120x^3}{7!} + \cdots$$

ie., $$-\frac{1}{y^2}y_3 + \frac{6}{y^3}y_1y_2 - \frac{6}{y^4}y_1^3 = \frac{x}{5} - \frac{x^3}{42} + \cdots \qquad \ldots (3)$$

At $\quad x = 0, \; y_3(0) = 0$

Differentiating (3) w.r.t. x we have, $\qquad \therefore \quad y_3(0) = 0$

$$-\frac{1}{y^2}y_4 + \frac{2}{y^3}y_1y_3 + \frac{6y_1}{y^3}y_3 + \frac{6y_2}{y^3}y_2 + 6y_1y_2\left(\frac{-3}{y^4}\right)y_1$$

$$-\frac{6}{y^4}(3y_1^2y_2) - 6y_1^3\left(\frac{-4}{y^5}\right)y_1 = \frac{1}{5} - \frac{x^2}{14} + \cdots$$

At $\quad x = 0 \;,\; -y_4(0) + 6(1/9) = 1/5$

or $\quad y_4(0) = 2/3 - 1/5 = 7/15 \qquad \therefore \quad y_4(0) = 7/15$

Substituting these values in the expansion of $y(x)$ we get,

$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$$

>> We have Maclaurin's series

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \qquad \ldots (1)$$

By data $f(x) = y(x) = \dfrac{x}{e^{x-1}}$; $y(0) = 0$

i.e. $\quad y = \dfrac{x}{e^x \cdot e^{-1}} = \dfrac{e\,x}{e^x}$ or $e^x y = e\,x$

We differentiate this equation successively four times and evaluate at $x = 0$ as follows

$e^x y_1 + e^x y = e \,;\, 1 \cdot y_1(0) + 1 \cdot 0 = e \qquad \therefore \quad y_1(0) = e$

$e^x y_2 + 2e^x y_1 + e^x y = 0 ; y_2(0) + 2e = 0 \qquad \therefore \quad y_2(0) = -2e$

$e^x y_3 + 3e^x y_2 + 3e^x y_1 + e^x y = 0 \,;\, y_3(0) - 6e + 3e = 0 \qquad \therefore \quad y_3(0) = 3e$

$e^x y_4 + 4e^x y_3 + 6e^x y_2 + 4e^x y_1 + e^x y = 0 \qquad \therefore \quad y_4(0) = -4e$

Thus by substituting these values in (1) we have,

$$\frac{x}{e^{x-1}} = e\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots\right)$$

Note : *If $y = f(x)$ is of the form $\dfrac{x}{e^x - 1}$ then $y(0)$ becomes $\dfrac{0}{0}$ which is indeterminate. We have to use the Maclaurin's series of e^x given by $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ so that*

$$y(x) = \frac{x}{x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \cdots} = \frac{1}{1 + \dfrac{x}{2} + \dfrac{x^2}{6} + \dfrac{x^3}{24} + \cdots}\ ;\ y(0) = 1$$

We consider $\dfrac{1}{y} = 1 + \dfrac{x}{2} + \dfrac{x^2}{6} + \dfrac{x^3}{24} + \cdots$ *and proceed to differentiate successively*
(Similar to the previous problem)

1. Expand $\tan x$ about the point $x = \pi/4$ upto the third degree terms and hence find $\tan 46°$
2. Show that
$\sqrt{x} = \sqrt{2}\,[\,1 + (x-2)/4 - (x-2)^2/32 + (x-2)^3/128 \ldots]$

Obtain the Maclaurin's expansion of the following functions as indicated

3. $e^{\cos x}$ upto the fourth degree terms
4. $\log(\sec x + \tan x)$ upto the first three non vanishing terms
5. $\log(1 + \sin^2 x)$ upto the fourth degree terms.
6. $\log(1 + \tan x)$ upto the third degree terms.
7. $e^x \sin x$ upto the fifth degree terms.
8. Show that : $\log(x + \sqrt{x^2+1}) = x - \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4}\dfrac{x^5}{5} - \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\dfrac{x^7}{7} + \cdots$
and hence find the value of $\log_e 2$ correct to two decimal places
9. Show that if $x = \sin t$, $y = \sin mt$ then
$y(x) = mx - \dfrac{m(m^2-1)}{3!}x^3 + \dfrac{m(m^2-1)(m^2-9)}{5!}x^5 - \cdots$
10. Show that $(\sin^{-1} x)^2 = \dfrac{2x^2}{2!} + \dfrac{2^2}{4!}\,2x^4 + \dfrac{2^2 \cdot 4^2}{6!}\,2x^6 + \cdots$

[**Hint** : *Use the n^{th} derivative approach for problems 8 to 10.*]

ANSWERS

1. $1+2(x-\pi/4)+2(x-\pi/4)^2+(8/3)(x-\pi/4)^3$; $1 \cdot 035$

3. $e\left[1-\frac{x^2}{2}+\frac{x^4}{6}\right]$

4. $x+x^3/6)+(x^5/24)$

5. $x^2-(5x^4/6)$

6. $x-(x^2/2)+(2x^3/3)$

7. $x+x^2+(x^3/3)-(x^5/30)$

Unit - II

2.1 Indeterminate forms

2.11 Introduction

If an expression $f(x)$ at $x = a$ assumes forms like $0/0$, ∞/∞ , $0 \times \infty$, $\infty - \infty$, 0^0 , ∞^0 , 1^∞ which do not represent any value are called *indeterminate forms*. The concept of limit gives a meaningful value for the function $f(x)$ at $x = a$ overcoming these indeterminate forms.

The reader is familiar with the evaluation of limit mostly in the cases of $0/0$ or ∞/∞ without the involvement of differentiation. Few more indeterminate forms $\infty - \infty$, $\infty \times 0$, ∞^0 , 0^0 , 1^∞ can be reduced to the two basic indeterminate forms $0/0$ and ∞/∞. Then limit is found passing through a process of differentiation warranted by a very simple rule called L' Hospital's (*French Mathematician*) rule which is established by using Cauchy's mean value theorem.

2.12 L' Hospital's rule (Theorem)

Statement : If $f(x)$ and $g(x)$ are two functions such that

(i) $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ *ie.,* $f(a) = 0 = g(a)$

(ii) $f'(x)$ and $g'(x)$ exist and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note : *Extension of the theorem*

If $f'(a) = 0$ and $g'(a) = 0$ then we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} \text{ and so on.}$$

Working procedure for problems by applying L' Hospital's rule

➲ The rule is applicable for the form $0/0$. It can also be applied for the form ∞/∞ as we can write

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left\{ \frac{1/g(x)}{1/f(x)} \right\}$$

where $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$

➲ However while applying the rule in this case also, we follow the usual procedure of differentiating the numerator $f(x)$ and the denominator $g(x)$ separately. If the indeterminate form persists after applying the rule once, we can apply the rule repeatedly till we arrive at a definite value. It is highly advisable to look for simplification at each stage. Problems have been bifurcated into four types and the procedure too has been explained separately in each type.

➲ The following four standard limits and well known simple properties connected with limits can be readily used.

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\lim_{x \to 0} \frac{x}{\sin x} = 1$ (iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$ (iv) $\lim_{x \to 0} \frac{x}{\tan x} = 1$

The rule can be applied directly in the case of forms $0/0$ and ∞/∞. In the cases of $\infty - \infty$ and $\infty \times 0$, we have to employ simple methods (*taking L.C.M., using equivalent trigonometric expressions etc.*) to simplify the given expression in bringing it to the form $0/0$ or ∞/∞ so that the L' Hospital's rule can be employed.

1. Let $k = \lim_{x \to 0} \dfrac{x e^x - \log(1+x)}{x^2} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospitals rule,

$$k = \lim_{x \to 0} \frac{x e^x + e^x - 1/1+x}{2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{x e^x + e^x + e^x + 1/(1+x)^2}{2} = \frac{0+1+1+1}{2} = \frac{3}{2}$$

Thus $\quad$ $\boldsymbol{k = 3/2}$

2. Let $k = \lim_{x \to 0} \dfrac{1 - \cos x}{x \log(1+x)} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$= \lim_{x \to 0} \frac{\sin x}{x \cdot (1/1 + x) + \log(1+x)} \quad \ldots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x}{x \cdot -1/(1+x)^2 + 1/1+x + 1/1+x} = \frac{1}{0+1+1} = \frac{1}{2}$$

Thus $\quad$ $\boldsymbol{k = 1/2}$

3. Let $k = \lim_{x \to \pi/2} \dfrac{\log(\sin x)}{(\pi/2 - x)^2} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule, $k = \lim_{x \to \pi/2} \dfrac{\cos x/\sin x}{-2(\pi/2 - x)}$

ie., $\quad = \lim_{x \to \pi/2} \dfrac{\cot x}{-2(\pi/2 - x)} \quad \ldots \left(\dfrac{0}{0}\right)$

Now $\quad k = \lim_{x \to \pi/2} \dfrac{-\text{cosec}^2 x}{2} = \dfrac{-1}{2}$

Thus $\quad$ $\boldsymbol{k = -1/2}$

4. Let $k = \lim_{x \to 0} \dfrac{a^x - b^x}{x} \quad \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{1} = \log a - \log b = \log(a/b)$$

Thus $\quad$ $\boldsymbol{k = \log(a/b)}$

5. Let $k = \lim\limits_{x \to 0} \dfrac{\sinh x - x}{\sin x - x \cos x} \cdots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{\cosh x - 1}{\cos x + x \sin x - \cos x} = \lim_{x \to 0} \frac{\cosh x - 1}{x \sin x} \cdots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \lim_{x \to 0} \frac{\sinh x}{x \cos x + \sin x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cosh x}{-x \sin x + \cos x + \cos x} = \frac{1}{2}$$

Thus $\quad k = 1/2$

(**Note** : *We have applied the rule thrice in this example*)

6. Let $k = \lim\limits_{x \to a} \dfrac{x^x - a^x}{x^a - a^a} \cdots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to a} \frac{x^x (1 + \log x) - a^x \log a}{a x^{a-1} - 0} \cdots \left(\frac{0}{0}\right)$$

$$k = \frac{a^a (1 + \log a) - a^a \log a}{a \cdot a^{a-1}} = \frac{a^a}{a^a} = 1$$

Thus $\quad k = 1$

7. Let $k = \lim\limits_{x \to \pi/2} \dfrac{\log (x - \pi/2)}{\tan x} \cdots \left(\dfrac{-\infty}{\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to \pi/2} \frac{1/(x - \pi/2)}{\sec^2 x} = \lim_{x \to \pi/2} \frac{\cos^2 x}{(x - \pi/2)} \cdots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \lim_{x \to \pi/2} \frac{2 \cos x \sin x}{1} = 0$$

Thus $\quad k = 0$

8. Let $k = \lim\limits_{x \to 0} \log_{\sin x} \sin 2x$

(We have the property $\log_b a = \log a / \log b$ for any base)

Now $k = \lim\limits_{x \to 0} \dfrac{\log(\sin 2x)}{\log(\sin x)} \quad \ldots \left(\dfrac{-\infty}{-\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{2\cos 2x/\sin 2x}{\cos x/\sin x} = \lim_{x \to 0} \frac{2\cot 2x}{\cot x} = \lim_{x \to 0} \frac{2\tan x}{\tan 2x} \quad \ldots \left(\frac{0}{0}\right)$$

$$\therefore \qquad k = \lim_{x \to 0} \frac{2\sec^2 x}{2\sec^2 2x} = 1.$$

Thus $\quad k = 1$

9. Let $k = \lim\limits_{x \to 0} \dfrac{\log x}{\operatorname{cosec} x} \quad \ldots \left(\dfrac{-\infty}{\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x} = -\lim_{x \to 0} \frac{\sin x}{x} \cdot \tan x$$

$$= -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \tan x = -1 \cdot 0 = 0$$

Thus $\quad k = 0$

Remark $\sin x \tan x / x$ *is of the form* $0/0$ *and the rule could have been applied again to obtain the answer. But it is always advisable to look for simplification and use standard limits at the right juncture.*

10. Let $k = \lim\limits_{x \to 0} \log_{\tan bx} \tan ax = \lim\limits_{x \to 0} \dfrac{\log(\tan ax)}{\log(\tan bx)} \quad \ldots \left(\dfrac{-\infty}{-\infty}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{a\sec^2 ax/\tan ax}{b\sec^2 bx/\tan bx} = \lim_{x \to 0} \frac{a\sec^2 ax \tan bx}{b\sec^2 bx \tan ax}$$

ie.,
$$= \frac{a}{b} \lim_{x \to 0} \frac{1}{\cos^2 ax} \cdot \frac{\cos ax}{\sin ax} \cdot \frac{\sin bx}{\cos bx} \cdot \cos^2 bx$$

$$= \frac{a}{b} \lim_{x \to 0} \frac{\sin bx \cos bx}{\sin ax \cos ax} = \frac{a}{b} \lim_{x \to 0} \frac{\sin 2bx}{\sin 2ax} \quad \ldots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \frac{a}{b} \lim_{x \to 0} \frac{2b \cos 2bx}{2a \cos 2ax} = 1$$

Thus $\quad \boldsymbol{k = 1}$

Note : *Alternative simplification after the first step*

$$k = \lim_{x \to 0} \frac{a \sec^2 ax}{b \sec^2 bx} \cdot \lim_{x \to 0} \frac{\tan bx}{\tan ax}$$

$$= \frac{a}{b} \lim_{x \to 0} \frac{\tan bx}{\tan ax} \cdots \left(\frac{0}{0}\right) \quad \text{since} \quad \sec 0 = 1$$

Now, applying the rule again we get

$$k = \frac{a}{b} \lim_{x \to 0} \frac{b \sec^2 bx}{a \sec^2 ax} = 1$$

Thus $\quad \boldsymbol{k = 1}$

11. Let $k = \lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x} \right] \cdots (\infty - \infty)$

We need to simplify the given expression.

ie., $\quad = \lim_{x \to 1} \left[\frac{x \log x - (x-1)}{(x-1) \log x} \right] \cdots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 1}\left[\frac{x \cdot 1/x + \log x - 1}{(x-1) \cdot 1/x + \log x}\right] = \lim_{x \to 1}\left[\frac{x \log x}{(x-1) + x \log x}\right] \cdots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \lim_{x \to 1} \frac{1 + \log x}{1 + (1 + \log x)} = \frac{1+0}{2+0} = \frac{1}{2}$$

Thus $\quad \boldsymbol{k = 1/2}$

12. $$k = \lim_{x \to 2}\left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right] \cdots (\infty - \infty)$$

ie., $$= \lim_{x \to 2}\left[\frac{\log(x-1) - (x-2)}{(x-2)\log(x-1)}\right] \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$= \lim_{x \to 2}\left[\frac{1/(x-1) - 1}{(x-2/x-1) + \log(x-1)}\right] \quad \text{We shall simplify}$$

ie., $$= \lim_{x \to 2}\left[\frac{2-x}{(x-2) + (x-1)\log(x-1)}\right] \cdots \left(\frac{0}{0}\right)$$

$$\therefore \quad k = \lim_{x \to 2}\left[\frac{-1}{1 + 1 + \log(x-1)}\right] = \frac{-1}{2}$$

Thus $\quad \boldsymbol{k = -1/2}$

13. Let $$k = \lim_{x \to 0}\left[\frac{a}{x} - \cot(x/a)\right] \quad (\infty - \infty)$$

ie., $$= \lim_{x \to 0}\left[\frac{a}{x} - \frac{\cos(x/a)}{\sin(x/a)}\right]$$

$$= \lim_{x \to 0}\left[\frac{a \sin(x/a) - x \cos(x/a)}{x \sin(x/a)}\right] \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0}\left[\frac{a \cdot 1/a \cdot \cos(x/a) + x \cdot 1/a \cdot \sin(x/a) - \cos(x/a)}{x \cdot 1/a \cdot \cos(x/a) + \sin(x/a)}\right]$$

ie., $$= \lim_{x \to 0}\left[\frac{x \sin(x/a)}{x \cos(x/a) + a \sin(x/a)}\right] \cdots \left(\frac{0}{0}\right)$$

$$k = \lim_{x \to 0} \left[\frac{x \cdot 1/a \cos(x/a) + \sin(x/a)}{x \cdot -1/a \cdot \sin(x/a) + \cos(x/a) + \cos(x/a)} \right]$$

ie., $$= \frac{0+0}{0+1+1} = \frac{0}{2} = 0$$

Thus $\boldsymbol{k = 0}$

14. Let $k = \lim_{x \to 0} \left[\frac{1}{x} - \frac{\log(1+x)}{x^2} \right] \ldots \left(\infty - \frac{0}{0} \right)$

ie., $$= \lim_{x \to 0} \frac{x - \log(1+x)}{x^2} \ldots \left(\frac{0}{0} \right)$$

Applying L'Hospital's rule,

$$k = \lim_{x \to 0} \frac{1 - (1/1+x)}{2x} \ldots \left(\frac{0}{0} \right)$$

$\therefore$ $$k = \lim_{x \to 0} \frac{1/(1+x)^2}{2} = \frac{1}{2}$$

Thus $\boldsymbol{k = 1/2}$

15. Let $k = \lim_{x \to \pi/2} (2x \tan x - \pi \sec x) \ldots (\infty - \infty)$

ie., $$= \lim_{x \to \pi/2} \left[2x \frac{\sin x}{\cos x} - \pi \frac{1}{\cos x} \right]$$

ie., $$= \lim_{x \to \pi/2} \left[\frac{2x \sin x - \pi}{\cos x} \right] \ldots \left(\frac{0}{0} \right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to \pi/2} \left| \frac{2x \cos x + 2 \sin x}{-\sin x} \right| = -2$$

Thus $\boldsymbol{k = -2}$

16. Let $k = \lim_{x \to 0} \tan x \cdot \log x \ldots (0 \times -\infty)$

ie., $$= \lim_{x \to 0} \frac{\log x}{\cot x} \ldots \left(\frac{-\infty}{\infty} \right)$$

Now applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x} = -\lim_{x \to 0} \frac{\sin^2 x}{x}$$

ie., $$= -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \sin x = -1 \cdot 0 = 0$$

Thus $\quad k = 0$

17. Let $k = \lim_{x \to \infty} x \tan(1/x) \quad (\infty \times 0)$

ie., $$= \lim_{x \to \infty} \frac{\tan(1/x)}{(1/x)} \quad \text{Put } 1/x = y \ ; \ y \to 0 \text{ as } x \to \infty$$

Hence $$k = \lim_{y \to 0} \frac{\tan y}{y} = 1$$

Thus $\quad k = 1$

18. Let $k = \lim_{x \to 1} (1 - x^2) \tan(\pi x/2) \ldots (0 \times \infty)$

ie., $$= \lim_{x \to 1} \frac{(1 - x^2)}{\cot(\pi x/2)} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 1} \frac{-2x}{-\pi/2 \operatorname{cosec}^2(\pi x/2)} = \frac{4}{\pi}$$

Thus $\quad k = 4/\pi$

19. Let $k = \lim_{x \to 0} x \log x \ldots (0 \times -\infty)$

ie., $$= \lim_{x \to 0} \frac{\log x}{(1/x)} \ldots \left(\frac{-\infty}{\infty}\right)$$

Now applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} -x = 0$$

Thus $\quad k = 0$

20. Let $k = \lim\limits_{x \to a} \log[2-(x/a)]\cot(x-a) \ldots (0 \times \infty)$

ie.,
$$= \lim_{x \to a} \frac{\log[2-(x/a)]}{\tan(x-a)} \ldots \left(\frac{0}{0}\right)$$

Now applying L' Hospital's rule,

$$k = \lim_{x \to a} \frac{1}{2-(x/a)} \cdot \frac{-1}{a} \cdot \frac{1}{\sec^2(x-a)} = -\frac{1}{a}$$

Thus $\quad \boldsymbol{k = -1/a}$

The given expression or its simplified form will be in the 0/0 form when $x = 0$ or as $x \to 0$ but will involve terms of the form $x^2 \sin x$, $x \sin^3 x$, $x \tan^2 x$ etc. In the event of applying the rule, the differentiation becomes tedious and we should not venture to do so. We can conveniently modify such terms so as to involve $(\sin x/x)^k$ or $(\tan x/x)^k$ or $(x/\sin x)^k$ or $(x/\tan x)^k$ which can be separated out from the given expression. These terms become 1 as $x \to 0$ with the result we will be left with a simple expression (*product gets eliminated*) in the 0/0 form for the application of L' Hospital's rule. Simplification at each step has to be explored.

21. Let $k = \lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x} \ldots \left(\dfrac{0}{0}\right)$

ie.,
$$= \lim_{x \to 0} \frac{\tan x - x}{x^2 \cdot \dfrac{\tan x}{x} \cdot x} = \lim_{x \to 0} \frac{\tan x - x}{x^3} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)$$

Hence $k = \lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3} \cdot 1 \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{\tan^2 x}{3x^2}$$

ie., $$= \frac{1}{3} \lim_{x \to 0} \left(\frac{\tan x}{x} \right)^2 = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

Thus $k = 1/3$

22. Let $k = \lim\limits_{x \to 0} \dfrac{x^2 + 2\cos x - 2}{x \sin^3 x} \ldots \left(\dfrac{0}{0} \right)$

ie., $$= \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \cdot \dfrac{\sin^3 x}{x^3} \cdot x^3} = \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4} \cdot \lim_{x \to 0} \left(\frac{x}{\sin x} \right)^3$$

Hence $k = \lim\limits_{x \to 0} \dfrac{x^2 + 2\cos x - 2}{x^4} \cdot 1 \ldots \left(\dfrac{0}{0} \right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{2x - 2\sin x}{4x^3} \ldots \left(\frac{0}{0} \right)$$

$$= \lim_{x \to 0} \frac{2 - 2\cos x}{12\, x^2} \ldots \left(\frac{0}{0} \right)$$

$$= \lim_{x \to 0} \frac{2\sin x}{24\, x} = \frac{1}{12} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{12} \cdot 1 = \frac{1}{12}$$

Thus $k = 1/12$

23. Let $k = \lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} \ldots \left(\dfrac{0}{0} \right)$

ie., $$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \cdot \dfrac{\sin x}{x} \cdot x}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \lim_{x \to 0} \frac{x}{\sin x}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot 1 \ldots \left(\frac{0}{0} \right)$$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{e^x + e^{-x} - (2/1+x)}{2x} \ldots \left(\frac{0}{0}\right)$$

$$k = \lim_{x \to 0} \frac{e^x - e^{-x} + 2/(1+x)^2}{2} = 1$$

Thus $k = 1$

24. Let $k = \lim_{x \to 0} \dfrac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x} \ldots \left(\dfrac{0}{0}\right)$

ie., $$= \lim_{x \to 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)^2$$

$$= \lim_{x \to 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \cdot 1 \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospitals rule,

$$k = \lim_{x \to 0} \frac{\cos x + \sin x - 1/1-x}{3x^2} \ldots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x + \cos x - 1/(1-x)^2}{6x} \ldots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\cos x - \sin x - 2/(1-x)^3}{6} = \frac{-3}{6} = \frac{-1}{2}$$

Thus $k = -1/2$

25. Let $k = \lim_{x \to 0} \left[\dfrac{1}{x^2} - \cot^2 x\right] \ldots (\infty - \infty)$

ie., $$= \lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\tan^2 x}\right] = \lim_{x \to 0} \left[\frac{\tan^2 x - x^2}{x^2 \tan^2 x}\right]$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)^2$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \cdot 1 \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x\to 0} \frac{2\tan x \sec^2 x - 2x}{4x^3} \dots \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2\tan x \cdot 2\sec^2 x \tan x + 2\sec^4 x - 2}{12x^2}$$

ie., $$= 2\lim_{x\to 0} \frac{2\sec^2 x \tan^2 x + \sec^4 x - 1}{12x^2} \dots \left(\frac{0}{0}\right)$$

(Further differentiation will be tedious and hence we simplify the term $\sec^4 x - 1$)

Now, $(\sec^4 x - 1) = (\sec^2 x - 1)\ (\sec^2 x + 1)$

ie., $(\sec^4 x - 1) = \tan^2 x(\sec^2 x + 1) = \sec^2 x \tan^2 x + \tan^2 x$

Hence $$k = \lim_{x\to 0} \frac{2\sec^2 x \tan^2 x + \sec^2 x \tan^2 x + \tan^2 x}{6x^2}$$

$$= \lim_{x\to 0} \frac{\tan^2 x\,(3\sec^2 x + 1)}{6x^2}$$

$$= \frac{1}{6} \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2 \cdot \lim_{x\to 0} (3\sec^2 x + 1)$$

$$= \frac{1}{6} \cdot 1 \cdot 4 = \frac{2}{3}$$

Thus $k = 2/3$

26. Let $$k = \lim_{x\to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] \qquad (\infty - \infty)$$

ie., $$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \cdot \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^2$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \cdot 1 \dots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$k = \lim_{x\to 0} \frac{2\sin x \cos x - 2x}{4x^3} = \lim_{x\to 0} \frac{\sin 2x - 2x}{4x^3} \dots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\cos 2x - 2}{12x^2} \text{ or } \lim_{x \to 0} \frac{\cos 2x - 1}{6x^2}$$

$$= \frac{1}{6} \lim_{x \to 0} \frac{-2\sin^2 x}{x^2} = \frac{-1}{3} \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^2 = \frac{-1}{3} \cdot 1 = -\frac{1}{3}$$

Thus $k = -1/3$

It is evident that the function involved will be of the form $[f(x)]^{g(x)}$ and we have to find the limit as $x \to a$.

Let $k = \lim_{x \to a} [f(x)]^{g(x)}$

Taking logarithms on both sides we have,

$$\log_e k = \lim_{x \to a} g(x) \cdot \log[f(x)]$$

We can evaluate the limit on the R.H.S as already discussed and let us suppose that the limit is equal to l.

ie., $\log_e k = l \Rightarrow k = e^l$ which is the required limit.

Remark : *One of the common question is that why 1^∞ is indeterminate ?*

Let $k = \lim_{x \to a} [f(x)]^{g(x)} \ldots 1^\infty$

$\Rightarrow$ $\log_e k = \lim_{x \to a} g(x) \log f(x) \ldots \infty \times \log 1 = \infty \times 0$

which is indeterminate. On the otherhand if $k = \lim_{x \to a} [f(x)]^{g(x)}$ is of the form c^∞ where $c \neq 1$ we have

$$\log_e k = \lim_{x \to a} g(x) \log f(x) = \infty \times \log c = \infty$$

27. Let $k = \lim\limits_{x \to 1} x^{1/1-x} \ldots (1^\infty)$

$$\Rightarrow \quad \log_e k = \lim_{x \to 1} \frac{1}{1-x} \log x = \lim_{x \to 1} \frac{\log x}{1-x} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 1} \frac{1/x}{-1} = -1$$

ie., $\quad \log_e k = -1$

Thus $\quad \boldsymbol{k = e^{-1} = 1/e}$

28. Let $k = \lim\limits_{x \to 0} (\cos x)^{1/x^2} \ldots (1^\infty)$

$$\Rightarrow \quad \log_e k = \lim_{x \to 0} \frac{\log(\cos x)}{x^2} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{-\sin x/\cos x}{2x} = \frac{-1}{2} \lim_{x \to 0} \frac{\tan x}{x} = -\frac{1}{2} \cdot 1$$

ie., $\quad \log_e k = -1/2 \Rightarrow k = e^{-1/2} = 1/\sqrt{e}$

Thus $\quad \boldsymbol{k = 1/\sqrt{e}}$

29. Let $k = \lim\limits_{x \to \pi/2} (\sin x)^{\tan x}$ (1^{∞})

$\Rightarrow \quad \log_e k = \lim\limits_{x \to \pi/2} \tan x \log(\sin x) \ldots (\infty \times 0)$

ie., $\quad = \lim\limits_{x \to \pi/2} \dfrac{\log(\sin x)}{\cot x} \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to \pi/2} \frac{\cos x/\sin x}{-\operatorname{cosec}^2 x} = \lim_{x \to \pi/2} -\sin x \cos x = 0$$

ie., $\quad \log_e k = 0$

Thus $\quad k = e^0 = 1$

30. Let $k = \lim\limits_{x \to a} \left[2-(x/a)\right]^{\tan(\pi x/2a)} \ldots (1^{\infty})$

$\Rightarrow \quad \log_e k = \lim\limits_{x \to a} \tan(\pi x/2a) \log\left[2-(x/a)\right]. \quad (\infty \times 0)$

ie., $\quad = \lim\limits_{x \to a} \dfrac{\log\left[2-(x/a)\right]}{\cot(\pi x/2a)} \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to a} \frac{\dfrac{1}{\left[2-(x/a)\right]} \times \dfrac{-1}{a}}{-\operatorname{cosec}^2(\pi x/2a) \times \pi/2a} = \frac{2}{\pi}$$

ie., $\quad \log_e k = 2/\pi$

Thus $\quad k = e^{2/\pi}$

31. Let $k = \lim\limits_{x \to 0} \left(\dfrac{a^x + b^x}{2}\right)^{1/x}$ (1^{∞})

$\Rightarrow \quad \log_e k = \lim\limits_{x \to 0} \dfrac{\log\{(a^x + b^x)/2\}}{x} \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{\dfrac{2}{a^x + b^x} \cdot \dfrac{1}{2}(a^x \log a + b^x \log b)}{1}$$

$$= \frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log \sqrt{ab}$$

ie., $\log_e k = \log \sqrt{ab}$

Thus $k = \sqrt{ab}$

32. Let $k = \lim_{x \to 0} [\sin^2(\pi/2 - x)]^{\sec^2(\pi/2 - x)}$ (1^∞)

Put $y = \pi/2 - x$ for convenience. As $x \to 0$, $y \to \pi/2$

Hence $k = \lim_{y \to \pi/2} (\sin^2 y)^{\sec^2 y}$ (1^∞)

$$\Rightarrow \quad \log_e k = \lim_{y \to \pi/2} \sec^2 y \cdot \log(\sin^2 y) \ldots (\infty \times 0)$$

$$= \lim_{y \to \pi/2} \frac{\log(\sin^2 y)}{\cos^2 y} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{y \to \pi/2} \frac{2 \sin y \cos y / \sin^2 y}{-2 \cos y \sin y} = -1$$

ie., $\log_e k = -1$

Thus $k = e^{-1} = 1/e$

33. Let $k = \lim_{x \to 1} (1 - x^2)^{1/\log(1 - x)} \ldots (0^0)$

$$\Rightarrow \quad \log_e k = \lim_{x \to 1} \frac{\log(1 - x^2)}{\log(1 - x)} \ldots \left(\frac{-\infty}{-\infty}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 1} \frac{-2x/1 - x^2}{-1/1 - x} = \lim_{x \to 1} \frac{2x(1 - x)}{(1 - x)(1 + x)}$$

ie., $\log_e k = \lim_{x \to 1} \frac{2x}{1 + x} = 1$

Thus $k = e^1 = e$

34. Let $k = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x \ldots (1^\infty)$

$\Rightarrow$ $\log_e k = \lim_{x \to \infty} x \log\left(1 + \frac{a}{x}\right) \ldots (\infty \times 0)$

$$= \lim_{x \to \infty} \frac{\log\left(1 + \frac{a}{x}\right)}{(1/x)} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to \infty} \frac{1\Big/\left(1 + \frac{a}{x}\right) \cdot -\frac{a}{x^2}}{(-1/x^2)} = a$$

ie., $\log_e k = a$

Thus $k = e^a$

35. Let $k = \lim_{x \to \infty} \left(\frac{ax + 1}{ax - 1}\right)^x$

We need to effect a basic simplification in this case of $x \to \infty$

ie., $k = \lim_{x \to \infty} \left[\frac{a + (1/x)}{a - (1/x)}\right]^x \ldots (1^\infty)$

Put $1/x = y$ for convenience, so that $y \to 0$ as $x \to \infty$

ie., $k = \lim_{y \to 0} \left(\frac{a + y}{a - y}\right)^{1/y} \ldots (1^\infty)$

$\Rightarrow$ $\log_e k = \lim_{y \to 0} \frac{1}{y} \log\left(\frac{a + y}{a - y}\right)$

$$= \lim_{y \to 0} \frac{\log(a + y) - \log(a - y)}{y} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{y \to 0} \frac{1/(a + y) + 1/(a - y)}{1} = \frac{1/a + 1/a}{1} = \frac{2}{a}$$

ie., $\log_e k = 2/a$

Thus $k = e^{2/a}$

36. Let $k = \lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)^{1/x^2} \quad \ldots (1^\infty)$

$$\Rightarrow \qquad \log_e k = \lim_{x \to 0} \frac{\log(\sin x/x)}{x^2} \ldots \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{\dfrac{1}{(\sin x/x)} \cdot \dfrac{x\cos x - \sin x}{x^2}}{2x}$$

$$\log_e k = \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^3}$$

$$= 1 \cdot \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^3} \ldots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-x\sin x + \cos x - \cos x}{6x^2}$$

ie., $$= \lim_{x \to 0} \frac{-1}{6} \cdot \frac{\sin x}{x} = \frac{-1}{6} \cdot 1 = \frac{-1}{6}$$

ie., $\log_e k = -1/6$

Thus $k = e^{-1/6}$

37. Let $k = \lim\limits_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{1/x} \quad \ldots (1^\infty)$

$$\Rightarrow \log_e k = \lim_{x \to 0} \frac{\log(\tan x/x)}{x} \ldots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{\dfrac{1}{(\tan x/x)} \cdot \dfrac{x\sec^2 x - \tan x}{x^2}}{1}$$

ie., $$= \lim_{x \to 0} \frac{x}{\tan x} \cdot \lim_{x \to 0} \frac{x\sec^2 x - \tan x}{x^2}$$

$$1 \cdot \lim_{x \to 0} \frac{x\sec^2 x - \tan x}{x^2} \ldots \left(\frac{0}{0}\right)$$

$$\log_e k = \lim_{x \to 0} \frac{x \cdot 2\sec^2 x \tan x + \sec^2 x - \sec^2 x}{2x}$$

$$= \lim_{x \to 0} \sec^2 x \cdot \tan x = 0$$

ie., $\log_e k = 0$

Thus $k = e^0 = 1$

38. Let $k = \lim_{x \to 0} x^{\sin x} \ldots (0^0)$

$\Rightarrow$ $\log_e k = \lim_{x \to 0} \sin x \log x \ldots (0 \times -\infty)$

$$= \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \ldots \left(\frac{-\infty}{\infty}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x} = -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \tan x$$

ie., $\log_e k = -1 \cdot 0 = 0$

Thus $k = e^0 = 1$

39. Let $k = \lim_{x \to 0} \left(\frac{1}{x}\right)^{2\sin x} \ldots (\infty^0)$

$\Rightarrow$ $\log_e k = \lim_{x \to 0} 2\sin x \cdot \log(1/x) = -\lim_{x \to 0} 2\sin x \cdot \log x$

$= -2\lim_{x \to 0} \sin x \log x = 0$ (*same as the previous example*)

$\therefore$ $\log_e k = 0$

Thus $k = e^0 = 1$

40. Let $k = \lim_{x \to 0} (\cot x)^{\tan x} \ldots (\infty^0)$

$\Rightarrow$ $\log_e k = \lim_{x \to 0} \tan x \cdot \log(\cot x) \ldots (0 \times \infty)$

$$= \lim_{x \to 0} \frac{\log(\cot x)}{\cot x} \dots \left(\frac{\infty}{\infty}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x \to 0} \frac{-\operatorname{cosec}^2 x/\cot x}{-\operatorname{cosec}^2 x} = \lim_{x \to 0} \tan x = 0$$

ie., $\log_e k = 0$

Thus $\mathbf{k = e^0 = 1}$

41. Let $k = \lim_{x \to \infty} (\pi/2 - \tan^{-1} x)^{1/x} \dots (0^0)$

ie., $k = \lim_{x \to \infty} (\cot^{-1} x)^{1/x} = \lim_{x \to \infty} [\tan^{-1}(1/x)]^{1/x}$

Put $1/x = y$. As $x \to \infty$, $y \to 0$

Hence $k = \lim_{y \to 0} (\tan^{-1} y)^y \dots (0^0)$

$\Rightarrow$ $$\log_e k = \lim_{y \to 0} y \log(\tan^{-1} y) \dots (0 \times -\infty)$$

ie., $$= \lim_{y \to 0} \frac{\log(\tan^{-1} y)}{(1/y)} \dots \left(\frac{\infty}{\infty}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{y \to 0} \frac{1/\tan^{-1} y \cdot 1/1 + y^2}{-1/y^2}$$

$$= \lim_{y \to 0} \frac{-y^2}{\tan^{-1} y \cdot (1 + y^2)} \dots \left(\frac{0}{0}\right)$$

$$\log_e k = \lim_{y \to 0} \frac{-2y}{\tan^{-1} y \cdot 2y + 1} = \frac{0}{0 + 1} = 0$$

ie., $\log_e k = 0$

Thus $\mathbf{k = e^0 = 1}$

Miscellaneous Examples

42. *Find the value of the constants a and b such that* lim [illegible]

>> Let $k = \lim\limits_{x \to 0} \dfrac{a \cosh x - b \cos x}{x^2} = \dfrac{a-b}{0}$

We must have $a - b = 0$ in order to apply the L' Hospital's rule,

Hence $k = \lim\limits_{x \to 0} \dfrac{a \sinh x + b \sin x}{2x} \ldots \left(\dfrac{0}{0}\right)$

$$k = \lim_{x \to 0} \frac{a \cosh x + b \cos x}{2} = \frac{a+b}{2}$$

But we must have $k = 1$

$\therefore \quad (a+b)/2 = 1$ or $a + b = 2$

By solving the equations, $a - b = 0$ and $a + b = 2$ we get $a = 1$, $b = 1$

Thus **$a = 1, b = 1$**

43. *Find the constants a, b, c such that* lim [illegible] *may be eq*[illegible]

>> Let $k = \lim\limits_{x \to 0} \dfrac{a e^{x} - b \cos x + c e^{-x}}{x \sin x}$

ie., $$= \lim_{x \to 0} \frac{a e^{x} - b \cos x + c e^{-x}}{x^2} \cdot \lim_{x \to 0} \frac{x}{\sin x}$$

$$= \lim_{x \to 0} \frac{a e^{x} - b \cos x + c e^{-x}}{x^2} \cdot 1 = \frac{a-b+c}{0}$$

Therefore we must have,

$$a - b + c = 0 \qquad \ldots (1)$$

With (1) we have 0/0 form Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{a e^{x} + b \sin x - c e^{-x}}{2x} = \frac{a-c}{0}$$

Again we must have $a - c = 0$ to get 0/0 form. That is

$$a = c \qquad \ldots (2)$$

Applying the rule again we have,

$$k = \lim_{x \to 0} \frac{a e^{x} + b \cos x + c e^{-x}}{2} = \frac{a+b+c}{2} \text{ and } k = 2 \text{ by data}$$

ie., $(a+b+c)/2 = 2$ and therefore

$$a+b+c=4 \qquad \ldots (3)$$

Since $c = a$, (1) and (3) becomes $2a-b=0$ and $2a+b=4$

By solving we get $a = 1$ and $b = 2$. Hence $c = 1$

Thus $\mathbf{a = 1,\ b = 2,\ c = 1}$

>> Let $k = \lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3} \ldots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{2 \cos 2x + a \cos x}{3x^2} = \frac{2+a}{0}$$

We must have $2+a=0$ or $a=-2$ for the $0/0$ form.

$$\therefore \quad k = \lim_{x \to 0} \frac{-4 \sin 2x - a \sin x}{6x} \ldots \left(\frac{0}{0}\right)$$

$$k = \lim_{x \to 0} \frac{-8 \cos 2x - a \cos x}{6} = \frac{-(8+a)}{6} = -1 \text{ since } a = -2$$

Thus $\mathbf{a = -2}$ and **finite limit = -1**

15. l

>> Let $k = \lim\limits_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x} \ldots \left(\dfrac{0}{0}\right)$

We note that $\lim\limits_{x \to 0} (1+x)^{1/x} = e$ and hence we have $0/0$ form

Applying L' Hospital's rule,

$$k = \lim_{x \to 0} \frac{\frac{d}{dx}[(1+x)^{1/x}]}{1} = \lim_{x \to 0} \frac{du}{dx} \qquad \ldots (1)$$

where $u = (1+x)^{1/x}$ and let us find $\dfrac{du}{dx}$

Now $\quad \log u = \dfrac{1}{x}\log(1+x) = \dfrac{\log(1+x)}{x}$

Differentiating w.r.t. x we get,

$$\frac{1}{u}\frac{du}{dx} = \frac{x \cdot \dfrac{1}{1+x} - \log(1+x)}{x^2}$$

$$\frac{du}{dx} = u\left\{\frac{x-(1+x)\log(1+x)}{(1+x)\,x^2}\right\}$$

Now $\quad \displaystyle\lim_{x\to 0}\frac{du}{dx} = \lim_{x\to 0} u \cdot \lim_{x\to 0}\left\{\frac{x-(1+x)\log(1+x)}{x^2+x^3}\right\}$

But $\quad \displaystyle\lim_{x\to 0} u = e.$ By using (1) we have

$$k = e\lim_{x\to 0}\left\{\frac{x-(1+x)\log(1+x)}{x^2+x^3}\right\} \quad \left(\frac{0}{0}\right)$$

Hence $\quad \displaystyle k = e\lim_{x\to 0}\left\{\frac{1-1-\log(1+x)}{2x+3x^2}\right\} \cdots \left(\frac{0}{0}\right)$

$$= e\lim_{x\to 0}\frac{-1/(1+x)}{2+6x} = e\cdot\frac{-1}{2} = \frac{-e}{2}$$

Thus $\quad \boldsymbol{k = -e/2}$

__Evaluate the following limits.__

1. $\displaystyle\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}$

2. $\displaystyle\lim_{x\to \pi/4}\frac{\sec^2 x - 2\tan x}{1+\cos 4x}$

3. $\displaystyle\lim_{x\to 0}\frac{x-\log(1+x)}{1-\cos x}$

4. $\displaystyle\lim_{x\to 0}\frac{\sin x - \sin^{-1} x}{x^2}$

5. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1+\sqrt{x-1}}{\sqrt{x^2-1}}$

6. $\displaystyle\lim_{x\to 0}\frac{\log(1+kx^2)}{1-\cos x}$

7. $\lim\limits_{x \to 1} \dfrac{\log(1-x)}{\cot \pi x}$

8. $\lim\limits_{x \to \infty} \dfrac{x \cos(1/x)}{1+x}$

9. $\lim\limits_{x \to 0} \log_x \tan x$

10. $\lim\limits_{x \to \pi/2} \dfrac{\log(\cos x)}{\tan x}$

11. $\lim\limits_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right]$

12. $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$

13. $\lim\limits_{x \to 1} \left[\dfrac{1}{\log x} - \dfrac{x}{\log x} \right]$

14. $\lim\limits_{x \to 0} x \log \tan x$

15. $\lim\limits_{x \to \pi/4} (\tan x)^{\tan 2x}$

16. $\lim\limits_{x \to 1} (2-x)^{\tan(\pi x/2)}$

17. $\lim\limits_{x \to 0} \left[\dfrac{a^x + b^x + c^x}{3} \right]^{1/x}$

18. $\lim\limits_{x \to 0} \left(\dfrac{\tan x}{x} \right)^{1/x^2}$

19. Find the constants a and b such that $\lim\limits_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$ may be equal to unity.

20. If $\lim\limits_{x \to 0} \dfrac{x(1 - a\cos x) + b\sin x}{x^2} = \dfrac{1}{3}$, show that the constants a and b satisfy the identity $a + b = 0$.

ANSWERS

1. $1/2$	2. $1/2$	3. 1	4. 1
5. $1/\sqrt{2}$	6. 2	7. 0	8. 1
9. 1	10. 0	11. $1/2$	12. 0
13. -1	14. 0	15. $1/e$	16. $e^{2/\pi}$
17. $(abc)^{1/3}$	18. $e^{1/3}$	19. $a = -5/2,\ b = -3/2$	

Polar Curves

2.21 Introduction

We are conversant in representing the position of a point $P(x, y)$ in the *cartesian system* and accordingly (x, y) are called cartesian coordinates

In this topic we discuss another important system to represent a point in a plane known as the *polar system*.

2.22 Polar Coordinates

Initial reference is chosen by spotting a point O in the plane called as the *pole*.

A line OL drawn through O is called the *initial line*. If P is any given point in the plane, join the points O and P with the result an angle is formed at O.

The length of OP denoted by r is called the *radius vector* of the point P and the angle LOP denoted by θ measured in the anticlockwise direction is called the *vectorial angle*.

The pair r and θ represented by $P = (r, \theta)$ or $P(r, \theta)$ are called as the *polar coordinates* of the point P.

It is evident that r is positive and θ lies between 0 and 2π according as the position of the point P in the four quadrants.

We now proceed to establish the *relationship between the cartesian coordinates (x, y) and the polar coordinates (r, θ)*.

Let (x, y) and (r, θ) respectively represent the cartesian and polar coordinates of any point P in the plane where the origin O is taken as the pole and the x axis is taken as the initial line

From the figure we have $OQ = x$, $PQ = y$.

Also from the right angled triangle OQP we have

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{r} \quad \therefore \quad x = r\cos\theta \qquad \ldots (1)$$

$$\sin\theta = \frac{QP}{OP} = \frac{y}{r} \quad \therefore \quad y = r\sin\theta \qquad \ldots (2)$$

Further squaring and adding (1) and (2) we get

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2 \cdot 1 = r^2$$

$$\therefore \qquad r = \sqrt{x^2 + y^2} \qquad \ldots(3)$$

Also dividing (2) by (1) we get

$$\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \quad ie., \quad \tan\theta = \frac{y}{x}$$

$$\therefore \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \ldots(4)$$

The relations (1) and (2) determine the cartesian coordinates in terms of polar coordinates whereas relations (3) and (4) determine the polar coordinates in terms of cartesian coordinates.

It is evident that r is a function of θ (r *depends on* θ) and the equation in the form

$$r = f(\theta) \text{ or } f(r, \theta) = c, \; c \text{ being a constant}$$

is called the equation of the curve in the polar form or simply a *polar curve*.

We now proceed to establish some results related to polar curves

Angle between radius vector and tangent

Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$.

$$\therefore \qquad X\hat{O}P = \theta \text{ and } OP = r.$$

Let PL be the tangent to the curve at P subtending an angle ψ with the positive direction of the initial line (x – *axis*) and ϕ be the angle between the radius vector OP and the tangent PL. That is $O\hat{P}L = \phi$.

From the figure we have

$$\psi = \phi + \theta$$

(*Recall from geometry that an exterior angle is equal to the sum of the interior opposite angles*)

$$\Rightarrow \qquad \tan\psi = \tan(\phi + \theta)$$

$$\text{or} \qquad \tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta} \qquad \ldots(1)$$

Let (x, y) be the cartesian coordinates of P so that we have,

$$x = r\cos\theta, \; y = r\sin\theta$$

Since r is a function of θ, we can as well regard these as parametric equations in terms of θ

We also know from the geometrical meaning of the derivative that

$$\tan\psi = \frac{dy}{dx} = \text{slope of the tangent } PL$$

ie., $\tan\psi = \dfrac{dy}{d\theta} \Big/ \dfrac{dx}{d\theta}$ since x and y are functions of θ.

ie., $$\tan\psi = \frac{\dfrac{d}{d\theta}(r\sin\theta)}{\dfrac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta} \text{ where } r' = \frac{dr}{d\theta}$$

(We try to correlate this expression with the already existing expression for $\tan\psi$ *in* (1) *Observe that the positive term in the denominator of* (1) *is equal to* 1)

Dividing both the numerator and denominator by $r'\cos\theta$ we have,

$$\tan\psi = \frac{\dfrac{r\cos\theta}{r'\cos\theta} + \dfrac{r'\sin\theta}{r'\cos\theta}}{\dfrac{-r\sin\theta}{r'\cos\theta} + \dfrac{r'\cos\theta}{r'\cos\theta}}$$

ie., $$\tan\psi = \frac{\dfrac{r}{r'} + \tan\theta}{1 - \dfrac{r}{r'}\tan\theta} \qquad \ldots (2)$$

Comparing equations (1) and (2) we get

$$\tan\phi = \frac{r}{r'} = \frac{r}{\left(\dfrac{dr}{d\theta}\right)} \quad \text{or} \quad \boxed{\tan\phi = r\left(\frac{d\theta}{dr}\right)}$$

Equivalently we can write it in the form

$$\frac{1}{\tan\phi} = \frac{1}{r}\left(\frac{dr}{d\theta}\right) \quad \text{or} \quad \boxed{\cot\phi = \frac{1}{r}\left(\frac{dr}{d\theta}\right)}$$

Note *A question format - Prove with usual notations* $\tan\phi = r\dfrac{d\theta}{dr}$

2.4 Length of the perpendicular from the pole to the tangent

Let O be the pole and OL be the initial line. Let $P(r, \theta)$ be any point on the curve and hence we have $OP = r$ and $L\hat{O}P = \theta$.

Draw $ON = p$ (*say*) perpendicular from the pole on the tangent at P and let ϕ be the angle made by the radius vector with the tangent.

From the figure $O\hat{N}P = 90^\circ$ and $L\hat{O}P = \theta$

Now from the right angled triangle ONP

$$\sin\phi = \frac{ON}{OP}$$

ie., $\quad \sin\phi = \dfrac{p}{r}$ or $\boxed{p = r\sin\phi}$

(*This expression is the basic expression for the length of the perpendicular p. We proceed to present the expression for p in terms of θ in two standard forms*)

We have $p = r\sin\phi$ ... (1)

and $\quad \cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$ ... (2)

Squaring equation (1) and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{1}{\sin^2\phi} \quad ie., \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

or $\quad \dfrac{1}{p^2} = \dfrac{1}{r^2}(1 + \cot^2\phi)$

Now using (2) we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

or $\quad \boxed{\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2}$ ... (3)

Further, let $\dfrac{1}{r} = u$

Differentiating *w.r.t.* θ we get,

$$-\frac{1}{r^2}\left(\frac{dr}{d\theta}\right) = \frac{du}{d\theta} \Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2, \text{ by squaring .}$$

Thus (3) now becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \qquad \ldots (4)$$

Note : *The usual format of the question is as follows.*

(*i*) *Prove with usual notations* $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

(*ii*) *Prove that for the curve* $r = f(\theta)$,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \text{ where } u = \frac{1}{r}$$

WORKED PROBLEMS

Angle of intersection of two polar curves

Basically we know that the angle of intersection of any two curves is equal to the angle between the tangents drawn at the point of intersection of the two curves

Let $r = f_1(\theta)$ and $r = f_2(\theta)$ be two curves intersecting at the point P.

Let PT_1 and PT_2 be the tangents drawn to the curves at the point P.

It can be seen from the figure that ϕ_1 is the angle between the radius vector OP and the tangent PT_1 and ϕ_2 is the angle made by the radius vector OP with PT_2. It can be clearly seen that the angle between the two tangents is equal to $\phi_2 - \phi_1$

$\therefore$ *the acute angle of the intersection of the curves is equal to* $|\phi_2 - \phi_1|$

If $|\phi_2 - \phi_1| = \pi/2$ then we say that the two curves intersect *orthogonally*

Further if $\phi_2 - \phi_1 = \dfrac{\pi}{2}$ then $\phi_2 = \dfrac{\pi}{2} + \phi_1$

$$\therefore \quad \tan\phi_2 = \tan\left(\frac{\pi}{2} + \phi_1\right) = -\cot\phi_1 = -\frac{1}{\tan\phi_1}$$

or $\quad \mathbf{\tan\phi_1 \cdot \tan\phi_2 = -1.}$

This result serves as an alternative condition for the orthogonality of two polar curves

Working procedure for problems

- Given the equation in the form $r = f(\theta)$ we prefer to take logarithms first on both sides of the equation and then differentiate *w.r.t* θ which always gives the term $\frac{1}{r}\frac{dr}{d\theta}$ being the derivative of $\log r$ *w.r.t* θ.
- We directly substitute $\cot \phi$ or $\cot \phi_1$, $\cot \phi_2$ as the case may be for the term $\frac{1}{r}\frac{dr}{d\theta}$
- We simplify R.H.S too and try to put it in terms of cotangent *ie*, "cot" so that we obtain ϕ or ϕ_1 and ϕ_2 as the case may be.
- $|\phi_2 - \phi_1|$ or $|\phi_1 - \phi_2|$ will give the angle of intersection
- If this contains θ then we have to find θ by solving the pair of equations to obtain the angle of intersection independent of θ.
- Suppose we are not able to obtain ϕ_1 and ϕ_2 explicitly then we have to write the expressions for $\tan \phi_1$, $\tan \phi_2$ and use the formula

$$\tan(\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2}$$

- If $\tan(\phi_1 - \phi_2) = \alpha$ (*say*) then the angle of intersection is equal to $\tan^{-1}(\alpha)$
- Also if $\tan \phi_1 \cdot \tan \phi_2 = -1$ then,

$$\tan(\phi_1 - \phi_2) = \infty \quad \Rightarrow \quad \phi_1 - \phi_2 = \pi/2$$

Note *The following allied and compound angles trigonometric formulae will have frequent reference in problems.*

1. $\sin(\pi/2 - \theta) = \cos\theta \qquad \cos(\pi/2 - \theta) = \sin\theta$

   $\tan(\pi/2 - \theta) = \cot\theta \qquad \cot(\pi/2 - \theta) = \tan\theta$

2. $\sin(\pi/2 + \theta) = \cos\theta \qquad \cos(\pi/2 + \theta) = -\sin\theta$

   $\tan(\pi/2 + \theta) = -\cot\theta \qquad \cot(\pi/2 + \theta) = -\tan\theta$

3. $\tan(\pi/4 + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}, \quad \cot(\pi/4 + \theta) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

Also we use the results :

$$1 + \cos\theta = 2\cos^2(\theta/2), \qquad 1 - \cos\theta = 2\sin^2(\theta/2),$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2), \qquad \cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$$

46. $r = a(1 - \cos\theta)$ 47. $r^2 \cos 2\theta$

48. $r^m = a^m(\cos m\theta + \sin m\theta)$ 49. $l/r = 1 + e\cos\theta$

46. $\quad r = a(1 - \cos\theta)$

Taking logarithms on both sides, $\log r = \log a + \log(1 - \cos\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1 - \cos\theta}$$

ie., $\quad \cot\phi = \dfrac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot\left(\theta/2\right)$

Thus $\quad \cot\phi = \cot(\theta/2) \Rightarrow \boldsymbol{\phi = \theta/2}$

47. $\quad r^2 \cos 2\theta = a^2$

Taking logarithms on both sides we have

$$2\log r + \log(\cos 2\theta) = 2\log a$$

Differentiating *w.r.t* θ we get,

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{(-2\sin 2\theta)}{\cos 2\theta} = 0$$

ie., $\quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \tan 2\theta$

or $\quad \cot\phi = \cot(\pi/2 - 2\theta) \quad \Rightarrow \quad \boldsymbol{\phi = \pi/2 - 2\theta}$

48. $\quad r^m = a^m(\cos m\theta + \sin m\theta)$

Taking logarithms on both sides we have, $m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{(-m\sin m\theta + m\cos m\theta)}{(\cos m\theta + \sin m\theta)}$$

ie., $\quad \dfrac{m}{r}\dfrac{dr}{d\theta} = \dfrac{m(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$

Thus $\quad \cot\phi = \dfrac{\cos m\theta(1 - \tan m\theta)}{\cos m\theta(1 + \tan m\theta)} = \dfrac{1 - \tan m\theta}{1 + \tan m\theta}$

or $\quad \cot\phi = \cot(\pi/4 + m\theta) \Rightarrow \boldsymbol{\phi = \pi/4 + m\theta}$

49. $l/r = 1 + e\cos\theta$

Taking logarithms on both sides we have,

$$\log l - \log r = \log(1 + e\cos\theta)$$

Differentiating *w.r.t* θ we get,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1 + e\cos\theta}$$

ie., $\cot\phi = \dfrac{e\sin\theta}{1 + e\cos\theta}$ *(This cannot be simplified)*

or $\tan\phi = \dfrac{1 + e\cos\theta}{e\sin\theta} \Rightarrow \boldsymbol{\phi = \tan^{-1}\left(\dfrac{1 + e\cos\theta}{e\sin\theta}\right)}$

50. $r = a(1 + \cos\theta)$ at $\theta = \pi$

51. $r\cos^2(\theta/2) = a$ at $\theta = 2\pi/3$

52. $2a/r = 1 - \cos\theta$ at $\theta = 2\pi/3$

53. $r = a(1 + \sin\theta)$ at $\theta = \pi/2$

50. $r = a(1 + \cos\theta)$

$\Rightarrow \log r = \log a + \log(1 + \cos\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1 + \cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

Thus $\cot\phi = \cot(\pi/2 + \theta/2) \Rightarrow \boldsymbol{\phi = \pi/2 + \theta/2}$

At $\theta = \pi/3$, $\phi = \pi/2 + \pi/6$ or $\boldsymbol{\phi = 2\pi/3 = 120^\circ}$

Also we have $\psi = \theta + \phi = \pi/3 + 2\pi/3 = \pi = 180^\circ$

$\therefore$ slope of the tangent $= \tan\psi = \tan 180^\circ = 0$

51. $r\cos^2(\theta/2) = a$

$\Rightarrow \log r + 2\log\cos(\theta/2) = \log a$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} + 2\cdot\frac{-1/2\cdot\sin(\theta/2)}{\cos(\theta/2)} = 0$$

$\therefore \quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \tan(\theta/2)$

ie., $\quad \cot\phi = \cot(\pi/2 - \theta/2) \quad \Rightarrow \quad \boldsymbol{\phi = \pi/2 - \theta/2}$

At $\quad \theta = 2\pi/3,\ \phi = \pi/2 - \pi/3 = \pi/6 = \mathbf{30^o}$

Also $\quad \psi = \theta + \phi = 2\pi/3 + \pi/6 = 5\pi/6 = 150^o$

$\therefore$ slope of the tangent $=$ $\tan\psi = \tan 150^o = \mathbf{-1/\sqrt{3}}$

$(\because \tan(150^o) = \tan(90^o + 60^o) = -\cot 60^o = -1/\sqrt{3})$

52. $\quad \mathbf{2a/r = 1 - \cos\theta}$

$\Rightarrow \quad \log 2a - \log r = \log(1 - \cos\theta)$

Differentiating *w.r.t* θ we get,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$$

ie., $\quad -\cot\phi = \cot(\theta/2)$ or $\cot(-\phi) = \cot\theta/2 \Rightarrow \boldsymbol{\phi = -\theta/2}$

At $\quad \theta = 2\pi/3,\ \boldsymbol{\phi = -\pi/3 = -60^o}$

Also $\quad \psi = \theta + \phi = 2\pi/3 - \pi/3 = \pi/3 = 60^o$

$\therefore$ slope of the tangent $=$ $\tan\psi = \tan(60^o) = \mathbf{\sqrt{3}}$

53. $\quad \boldsymbol{r = a(1 + \sin\theta)}$

$\Rightarrow \quad \log r = \log a + \log(1 + \sin\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta} \quad ie., \quad \cot\phi = \frac{\cos\theta}{1+\sin\theta}$$

At $\quad \theta = \pi/2,\ \cot\phi = \dfrac{0}{1+1} = 0 \ \therefore\ \cot\phi = 0 \Rightarrow \boldsymbol{\phi = \pi/2}$

Also $\quad \psi = \theta + \phi = \pi/2 + \pi/2 = \pi$

slope of the tangent $=$ $\tan\psi = \tan\pi = \mathbf{0}$

Note *We can simplify R.H.S. and explicitly obtain* ϕ *as shown below*

$$\cot\phi = \frac{\cos^2(\theta/2) - \sin^2(\theta/2)}{\cos^2(\theta/2) + \sin^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)}$$

$$\cot\phi = \frac{[\cos(\theta/2)-\sin(\theta/2)]\,[\cos(\theta/2)+\sin(\theta/2)]}{[\cos(\theta/2)+\sin(\theta/2)]^2}$$

$$= \frac{[\cos(\theta/2)-\sin(\theta/2)]}{[\cos(\theta/2)+\sin(\theta/2)]} = \frac{\cos(\theta/2)\,[1-\tan(\theta/2)]}{\cos(\theta/2)\,[1+\tan(\theta/2)]}$$

ie., $$\cot\phi = \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$$

Thus $\cot\phi = \cot(\pi/4+\theta/2) \Rightarrow \phi = \pi/4+\theta/2$

If we put $\theta = \pi/2$ we obtain $\phi = \pi/4+\pi/4 = \pi/2$

Show that the following pairs of curves intersect

54. $r = a(1+\cos\theta)$

54. $\boldsymbol{r = a(1+\cos\theta)} \quad : \quad \boldsymbol{r = b(1-\cos\theta)}$

$\Rightarrow \log r = \log a + \log(1+\cos\theta) \quad : \quad \log r = \log b + \log(1-\cos\theta)$

Differentiating these w.r.t θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0+\frac{-\sin\theta}{1+\cos\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = 0+\frac{\sin\theta}{1-\cos\theta}$$

$$\cot\phi_1 = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} \quad : \quad \cot\phi_2 = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

ie., $\cot\phi_1 = -\tan(\theta/2) = \cot(\pi/2+\theta/2) \quad : \quad \cot\phi_2 = \cot(\theta/2)$

$\Rightarrow \quad \phi_1 = \pi/2+\theta/2 \quad : \quad \phi_2 = \theta/2$

$\therefore$ angle of intersection $= |\phi_1-\phi_2| = |\pi/2+\theta/2-\theta/2| = \pi/2$

Hence the curves intersect orthogonally.

55. $r = a(1+\sin\theta)\ :\ r = a(1-\sin\theta)$

$\Rightarrow$ $\log r = \log a + \log(1+\sin\theta)\ :\ \log r = \log a + \log(1-\sin\theta)$

Differentiating these *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-\cos\theta}{1-\sin\theta}$$

ie., $$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta} \quad : \quad \cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$$

(*Refering to the note in Ex-53, it requires quite a number of steps to obtain ϕ_1 and ϕ_2 explicitly in order to find $|\phi_1 - \phi_2|$. But it will suffice if we can show that* $\tan\phi_1 \cdot \tan\phi_2 = -1$)

We have $\tan\phi_1 = \dfrac{1+\sin\theta}{\cos\theta}$ and $\tan\phi_2 = \dfrac{1-\sin\theta}{-\cos\theta}$

$\therefore$ $$\tan\phi_1 \cdot \tan\phi_2 = \frac{1-\sin^2\theta}{-\cos^2\theta} = \frac{\cos^2\theta}{-\cos^2\theta} = -1$$

Hence the curves intersect orthogonally.

56. $r^n = a^n \cos n\theta \quad : \quad r^n = b^n \sin n\theta$

Taking logarithms we have,

$$n\log r = n\log a + \log(\cos n\theta) \ :\ n\log r = n\log b + \log(\sin n\theta)$$

Differentiating these *w.r.t* θ we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} \quad : \quad \frac{n}{r}\frac{dr}{d\theta} = \frac{n\cos n\theta}{\sin n\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

ie., $$\cot\phi_1 = \cot(\pi/2 + n\theta) \quad : \quad \cot\phi_2 = \cot n\theta$$

$\Rightarrow$ $$\phi_1 = \pi/2 + n\theta \quad : \quad \phi_2 = n\theta$$

$\therefore$ $$|\phi_1 - \phi_2| = |\pi/2 + n\theta - n\theta| = \pi/2$$

Hence the curves intersect orthogonally.

57. $\quad r^2 \sin 2\theta = a^2 \quad : \quad r^2 \cos 2\theta = b^2$

Taking logarithms we have,

$$2 \log r + \log(\sin 2\theta) = 2\log a \;:\; 2\log r + \log(\cos 2\theta) = 2\log b$$

Differentiating these *w.r.t* θ we get,

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0 \quad : \quad \frac{2}{r}\frac{dr}{d\theta} - \frac{2\sin 2\theta}{\cos 2\theta} = 0$$

ie., $\quad \dfrac{1}{r}\dfrac{dr}{d\theta} = -\cot 2\theta \quad : \quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \tan 2\theta$

ie., $\quad \cot\phi_1 = -\cot 2\theta \quad : \quad \cot\phi_2 = \tan 2\theta$

ie., $\quad \cot\phi_1 = \cot(-2\theta) \quad : \quad \cot\phi_2 = \cot(\pi/2 - 2\theta)$

$\Rightarrow \quad \phi_1 = -2\theta \quad : \quad \phi_2 = \pi/2 - 2\theta$

$\therefore \quad |\phi_1 - \phi_2| = |-2\theta - \pi/2 + 2\theta| = \pi/2$

Hence the curves intersect orthogonally.

58. $\quad r = 4\sec^2(\theta/2) \quad : \quad r = 9\,\text{cosec}^2(\theta/2)$

Taking logarithms we have,

$$\log r = \log 4 + 2\log\sec(\theta/2) \;:\; \log r = \log 9 + 2\log \text{cosec}(\theta/2)$$

Differentiating these *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2}{\sec(\theta/2)} \cdot \sec(\theta/2)\tan(\theta/2) \cdot \frac{1}{2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\,\text{cosec}(\theta/2)\cot(\theta/2)}{\text{cosec}(\theta/2)} \cdot \frac{1}{2}$$

ie., $\quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \tan(\theta/2) \quad : \quad \dfrac{1}{r}\dfrac{dr}{d\theta} = -\cot(\theta/2)$

ie., $\quad \cot\phi_1 = \cot(\pi/2 - \theta/2) \quad : \quad \cot\phi_2 = \cot(-\theta/2)$

$\Rightarrow \quad \phi_1 = \pi/2 - \theta/2 \quad : \quad \phi_2 = -\theta/2$

$\therefore \quad |\phi_1 - \phi_2| = |\pi/2 - \theta/2 + \theta/2| = \pi/2$

Hence the curves intersect orthogonally.

59. $r = ae^{\theta}$: $re^{\theta} = b$

$\Rightarrow$ $\log r = \log a + \theta \log e$: $\log r + \theta \log e = \log b$

But $\log e = 1$ Differentiating these *w.r.t* θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1 \quad : \quad \frac{1}{r}\frac{dr}{d\theta} + 1 = 0$$

ie., $\cot\phi_1 = 1$: $\cot\phi_2 = -1$

$\Rightarrow$ $\phi_1 = \pi/4$: $\phi_2 = -\pi/4$ or $3\pi/4$

$\therefore$ $|\phi_1 - \phi_2| = |\pi/4 + \pi/4| = \pi/2$

Hence the curves intersect orthogonally.

60. $r = \sin\theta + \cos\theta$: $r = 2\sin\theta$

$\Rightarrow$ $\log r = \log(\sin\theta + \cos\theta)$: $\log r = \log 2 + \log(\sin\theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta}$$

ie., $\cot\phi_1 = \dfrac{\cos\theta(1 - \tan\theta)}{\cos\theta(1 + \tan\theta)}$: $\cot\phi_2 = \cot\theta \Rightarrow \phi_2 = \theta$

ie., $\cot\phi_1 = \cot(\pi/4 + \theta) \Rightarrow \phi_1 = \pi/4 + \theta$

$\therefore$ $|\phi_1 - \phi_2| = |\pi/4 + \theta - \theta| = \pi/4$

The angle of intersection is $\pi/4$

61. $r = a \log \theta \quad : \quad r = a/\log \theta$

$\Rightarrow \quad \log r = \log a + \log(\log \theta) \quad : \quad \log r = \log a - \log(\log \theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\log\theta \cdot \theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\log\theta \cdot \theta}$$

ie., $$\cot\phi_1 = \frac{1}{\theta\log\theta} \quad : \quad \cot\phi_2 = -\frac{1}{\theta\log\theta}$$

Note : *we cannot find* ϕ_1 *and* ϕ_2 *explicitly.*

$\therefore \quad \tan\phi_1 = \theta\log\theta \quad : \quad \tan\phi_2 = -\theta\log\theta$

Now consider, $\tan(\phi_1 - \phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\tan\phi_2}$

ie., $$\tan(\phi_1 - \phi_2) = \frac{2\theta\log\theta}{1-(\theta\log\theta)^2} \qquad \ldots(1)$$

We have to find θ by solving the given pair of equations :
$r = a\log\theta$ and $r = a/\log\theta$.

Equating the R.H.S we have $a\log\theta = \dfrac{a}{\log\theta}$

ie., $(\log\theta)^2 = 1$ or $\log\theta = 1 \Rightarrow \theta = e$

Substituting $\theta = e$ in (1) we get,

$$\tan(\phi_1 - \phi_2) = \frac{2e}{1-e^2} \quad (\because \log e = 1)$$

$\therefore$ **angle of intersection** $= \phi_1 - \phi_2 = \tan^{-1}\left(\dfrac{2e}{1-e^2}\right) = 2\tan^{-1} e$

62. $r^2 \sin 2\theta = 4 \quad : \quad r^2 = 16\sin 2\theta$

$2\log r + \log(\sin 2\theta) = \log 4 \quad : \quad 2\log r = \log 16 + \log(\sin 2\theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0 \quad : \quad \frac{2}{r}\frac{dr}{d\theta} = \frac{2\cos 2\theta}{\sin 2\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \cot 2\theta$$

ie., $\cot\phi_1 = \cot(-2\theta)$: $\cot\phi_2 = \cot 2\theta$

$\Rightarrow$ $\phi_1 = -2\theta$: $\phi_2 = 2\theta$

$\therefore$ $|\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta$ ...(1)

Now consider $r^2 = \dfrac{4}{\sin 2\theta}$ and $r^2 = 16\sin 2\theta$

$\therefore$ $\dfrac{4}{\sin 2\theta} = 16\sin 2\theta$ or $4\sin^2 2\theta = 1$

ie., $\sin^2 2\theta = 1/4$ or $\sin 2\theta = 1/2 \Rightarrow 2\theta = \pi/6$ $\therefore$ $\theta = \pi/12$

Substituting $\theta = \pi/12$ in (1) we get $|\phi_1 - \phi_2| = \pi/3$

$\therefore$ **angle of intersection $= \pi/3 = 60^o$**

63. $r = a(1-\cos\theta)$: $r = 2a\cos\theta$

Taking logarithms we have,

$\log r = \log a + \log(1-\cos\theta)$: $\log r = \log 2a + \log(\cos\theta)$

Differentiating these *w.r.t* θ, we get,

$\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{1-\cos\theta}$: $\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{-\sin\theta}{\cos\theta}$

ie., $\cot\phi_1 = \dfrac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$: $\cot\phi_2 = -\tan\theta$

ie., $\cot\phi_1 = \cot(\theta/2)$: $\cot\phi_2 = \cot(\pi/2+\theta)$

$\Rightarrow$ $\phi_1 = \theta/2$: $\phi_2 = \pi/2+\theta$

$\therefore$ $|\phi_1 - \phi_2| = |\theta/2 - \pi/2 - \theta| = \pi/2 + \theta/2$ ...(1)

Now consider $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$\therefore$ $a(1-\cos\theta) = 2a\cos\theta$

or $3\cos\theta = 1$ or $\theta = \cos^{-1}(1/3)$

Substituting this value in (1) we get,

the angle of intersection $= \pi/2 + 1/2 \cdot \cos^{-1}(1/3)$

64. $r = 6\cos\theta \quad : \quad r = 2(1+\cos\theta)$

$\Rightarrow \quad \log r = \log 6 + \log(\cos\theta) \quad : \quad \log r = \log 2 + \log(1+\cos\theta)$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$$

ie., $\cot\phi_1 = -\tan\theta \quad : \quad \cot\phi_2 = \dfrac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$

ie., $\cot\phi_1 = \cot(\pi/2+\theta) \quad : \quad \cot\phi_2 = -\tan(\theta/2) = \cot(\pi/2+\theta/2)$

$\Rightarrow \quad \phi_1 = \pi/2+\theta \quad : \quad \phi_2 = \pi/2+\theta/2$

$\therefore \quad |\phi_1-\phi_2| = \theta/2$ ...(1)

Equating the R.H.S of the given equations we have

$$6\cos\theta = 2(1+\cos\theta) \text{ or } \cos\theta = 1/2 \Rightarrow \theta = \pi/3$$

$\therefore$ from (1) $|\phi_1-\phi_2| = \pi/6 = 30^\circ$

Hence the angle of intersection = $\pi/6 = 30^\circ$

65. $r^n = a^n\sec(n\theta+\alpha) \quad : \quad r^n = b^n\sec(n\theta+\beta)$

Taking logarithms we have,

$$n\log r = n\log a + \log\sec(n\theta+\alpha) : n\log r = n\log b + \log\sec(n\theta+\beta)$$

Differentiating these *w.r.t* θ, we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n\sec(n\theta+\alpha)\tan(n\theta+\alpha)}{\sec(n\theta+\alpha)} \quad ; \quad \frac{n}{r}\frac{dr}{d\theta} = \frac{n\sec(n\theta+\beta)\tan(n\theta+\beta)}{\sec(n\theta+\beta)}$$

ie., $\dfrac{1}{r}\dfrac{dr}{d\theta} = \tan(n\theta+\alpha) \quad : \quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \tan(n\theta+\beta)$

ie., $\cot\phi_1 = \cot[\pi/2-(n\theta+\alpha)] \quad : \quad \cot\phi_2 = \cot[\pi/2-(n\theta+\beta)]$

$\Rightarrow \quad \phi_1 = \pi/2-n\theta-\alpha \quad : \quad \phi_2 = \pi/2-n\theta-\beta$

$\therefore \quad |\phi_1-\phi_2| = |-\alpha+\beta| = \alpha-\beta$, where $\alpha > \beta$

Hence the angle of intersection $= \alpha-\beta$, where $\alpha > \beta$

66. $r = a(1+\cos\theta)$: $r^2 = a^2\cos 2\theta$

Taking logarithms we have,

$$\log r = \log a + \log(1+\cos\theta) \quad : \quad 2\log r = 2\log a + \log(\cos 2\theta)$$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} \quad : \quad \frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta}$$

ie., $\cot\phi_1 = \dfrac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$: $\cot\phi_2 = -\tan 2\theta$

ie., $\cot\phi_1 = -\tan(\theta/2)$: $\cot\phi_2 = \cot(\pi/2+2\theta)$

ie., $\cot\phi_1 = \cot(\pi/2+\theta/2)$: $\Rightarrow \phi_2 = \pi/2+2\theta$

$\Rightarrow \phi_1 = \pi/2+\theta/2$

$\therefore |\phi_1-\phi_2| = |\pi/2+\theta/2-\pi/2-2\theta| = 3\theta/2$ . . . (1)

Now, squaring the first of the given equations and then equating the R.H.S of the two equations we have

$$a^2(1+\cos\theta)^2 = a^2\cos 2\theta$$

ie., $1+2\cos\theta+\cos^2\theta = 2\cos^2\theta - 1$

or $\cos^2\theta - 2\cos\theta - 2 = 0$

$\therefore \cos\theta = \dfrac{2\pm\sqrt{4+8}}{2} = \dfrac{2\pm 2\sqrt{3}}{2} = 1\pm\sqrt{3}$

Since $\cos\theta$ cannot exceed 1 numerically we have to take

$$\cos\theta = 1-\sqrt{3} \quad\Rightarrow\quad \theta = \cos^{-1}(1-\sqrt{3})$$

Otherwise we can also have,

$$1-2\sin^2(\theta/2) = 1-\sqrt{3} \text{ or } \sin^2(\theta/2) = \sqrt{3}/2 = \sqrt{3/4}$$

ie., $\sin\left(\theta/2\right) = (\sqrt{3/4})^{1/2} = (3/4)^{1/4}$

$\therefore \theta/2 = \sin^{-1}\left(3/4\right)^{1/4}$

Substituting this value in (1) we get,

the angle of intersection $= 3\sin^{-1}(3/4)^{1/4}$

67. $r = a\theta/1+\theta \quad : \quad r = a/1+\theta^2$

Taking logarithms we have,

$$\log r = \log a + \log\theta - \log(1+\theta) \quad : \quad \log r = \log a - \log(1+\theta^2)$$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} - \frac{1}{1+\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-2\theta}{1+\theta^2}$$

ie., $$\cot\phi_1 = \frac{1}{\theta(1+\theta)} \quad : \quad \cot\phi_2 = \frac{-2\theta}{1+\theta^2}$$

$$\Rightarrow \quad \tan\phi_1 = \theta+\theta^2 \quad : \quad \tan\phi_2 = \frac{1+\theta^2}{-2\theta}$$

Also by equating the R.H.S of the given equations we have,

$$\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$$

or $\theta+\theta^3 = 1+\theta$ or $\theta^3 = 1 \Rightarrow \theta = 1$

$\therefore \quad \tan\phi_1 = 2$ and $\tan\phi_2 = -1$ at $\theta = 1$

Consider, $$\tan(\phi_1-\phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}$$

$$\therefore \quad \tan(\phi_1-\phi_2) = \frac{2-(-1)}{1+(-2)} = -3$$

Taking the absolute value, **the angle of intersection** $= \tan^{-1}(3)$

---

68. $r = a\theta \quad : \quad r = a/\theta$

$$\Rightarrow \quad \log r = \log a + \log\theta \quad : \quad \log r = \log a - \log\theta$$

Differentiating these *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} \quad : \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta}$$

ie., $$\cot\phi_1 = \frac{1}{\theta} \quad : \quad \cot\phi_2 = -\frac{1}{\theta}$$

or $$\tan\phi_1 = \theta \quad : \quad \tan\phi_2 = -\theta$$

Also by equating the R.H.S of the given equations we have

$$a\theta = a/\theta \text{ or } \theta^2 = 1 \Rightarrow \theta = \pm 1$$

When $\theta = 1,\ \tan\phi_1 = 1,\ \tan\phi_2 = -1$ and

when $\theta = -1,\ \tan\phi_1 = -1,\ \tan\phi_2 = 1.$

$$\therefore \quad \tan\phi_1 \cdot \tan\phi_2 = -1 \Rightarrow \phi_1 - \phi_2 = \pi/2$$

The curves intersect at right angles.

---

69. [illegible] $r^2 = a^2\cos 2\theta$ *with the initial line*

>> We have $r^2 = a^2\cos 2\theta$

$$\Rightarrow \quad 2\log r = 2\log a + \log(\cos 2\theta)$$

Differentiating *w.r.t* θ, we have,

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$$

ie., $\quad \cot\phi = \cot(\pi/2 + 2\theta) \Rightarrow \phi = \pi/2 + 2\theta$

If ψ is the angle made by the tangent with the initial line, $\psi - (\pi/2)$ will be the angle made by the normal with the initial line.

We know that $\psi = \phi + \theta = (\pi/2 + 2\theta) + \theta = \pi/2 + 3\theta$

Hence $\psi = \pi/2 + 3\theta \quad \Rightarrow \quad \psi - (\pi/2) = 3\theta$

Thus $(\pi/2) + 3\theta$ and 3θ are respectively the angles made by the tangent and the normal with the initial line.

---

70. [illegible] $\theta = 2\pi/3$ *are respectively parallel and perpendicular to the initial line*

>> We have $r = a(1 + \cos\theta)$

$$\Rightarrow \quad \log r = \log a + \log(1 + \cos\theta)$$

Differentiating *w.r.t* θ, we have,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

ie., $\quad \cot\phi = \cot(\pi/2 + \theta/2) \Rightarrow \phi = \pi/2 + \theta/2$

If ψ is the angle made by the tangent with the initial line then,

$$\psi = \phi + \theta = \pi/2 + 3\theta/2$$

At $\quad \theta = \pi/3 : \psi = \pi/2 + \pi/2 = \pi = 180^\circ$

At $\quad \theta = 2\pi/3 \; : \; \psi = \pi/2 + \pi = 3\pi/2 = 270^{\circ}$

slope of the tangents $\tan\psi$ are respectively 0 and ∞

Hence we conclude that the tangents to the given curve at $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively **parallel and perpendicular to the initial line.**

---

71. *For a given curve in the cartesian form prove that*

$$\tan\phi = \frac{x\,y' - y}{x + y\,y'}$$

>> We have with usual notations $\psi = \theta + \phi$

$\therefore \quad \phi = \psi - \theta \Rightarrow \tan\phi = \tan(\psi - \theta)$

or $$\tan\phi = \frac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta} \qquad \ldots(1)$$

We also have $\tan\psi = \dfrac{dy}{dx} = y'$ and

$x = r\cos\theta,\; y = r\sin\theta$ gives $\tan\theta = (y/x)$

Substituting these in (1) we get,

$$\tan\phi = \frac{y' - (y/x)}{1 + y'(y/x)} = \frac{x\,y' - y}{x + y\,y'}$$

Thus $$\tan\phi = \frac{x\,y' - y}{x + y\,y'}$$

Pedal equation of a polar curve

In the context of deriving an expression for the length of the perpendicular (p) from the pole to the tangent we obtained the expression in the form $p = r\sin\phi$

The equation of the given curve $r = f(\theta)$ expressed in terms of p and r is called as the pedal equation or $p - r$ equation of the curve $r = f(\theta)$

Remark : *Many equations of the standard cartesian curves $y = f(x)$ are expressible in the parametric form $x = f_1(t)$, $y = f_2(t)$. Eliminating t we get $y = f(x)$. We have a similar concept in respect of $r = f(\theta)$.*

Working procedure for finding the pedal equation of a polar curve

- Given $r = f(\theta)$ we first obtain ϕ.
- We substitute ϕ (*usually a function of* θ) into the equation $p = r\sin\phi$ so that this equation assumes the form $p = r\,g(\theta)$
- We need to eliminate θ between the equations :

$$r = f(\theta) \qquad \ldots(1)$$

$$p = r\,g(\theta) \qquad \ldots(2)$$

This will give us an equation in p and r being the required pedal equation.

➲ It may be noted that if we are unable to obtain ϕ explicitly in terms of θ, we have to sqaure and take the reciprocal of $p = r\sin\phi$.

This will give us :

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2\phi]$$

We substitute for $\cot\phi$ itself in terms of θ Elimination of θ by using the given equation will give us the pedal equation.

Find the pedal equation of the following curves .

72. $2a/r = (1+\cos\theta)$
73. $r(1-\cos\theta) = 2a$
74. $r^2 = a^2\sec 2\theta$
75. $r^n = a^n\cos n\theta$
76. $r^m = a^m(\cos m\theta + \sin m\theta)$
77. $r = 2(1+\cos\theta)$
78. $l/r = 1 + e\cos\theta$
79. $r^n = a^n\operatorname{sech} n\theta$

72. $\dfrac{2a}{r} = 1+\cos\theta$

$\Rightarrow \quad \log 2a - \log r = \log(1+\cos\theta)$

Differentiating *w.r.t* θ, we get,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

ie., $\quad \cot\phi = \cot(\pi/2-\theta/2) \Rightarrow \phi = \pi/2-\theta/2$

Consider $p = r\sin\phi$ and substituting the value of ϕ we have

$$p = r\sin(\pi/2-\theta/2) = r\cos(\theta/2)$$

Now we have $\dfrac{2a}{r} = 1+\cos\theta$ ... (1)

$$p = r\cos(\theta/2) \quad \text{... (2)}$$

We have to eliminate θ from (1) and (2)

(It will be convenient for elimination if we can have similar functions of θ in the R.H.S of the two equations)

(1) can be put in the form $\dfrac{2a}{r} = 2\cos^2(\theta/2)$ or $\dfrac{a}{r} = \cos^2(\theta/2)$

Also from (2), $\dfrac{p}{r} = \cos(\theta/2)$

Hence we get, $\frac{a}{r} = \left(\frac{p}{r}\right)^2$ or $\frac{a}{r} = \frac{p^2}{r^2}$ or $p^2 = ar$

Thus $p^2 = ar$ is the required pedal equation.

73. $r(1-\cos\theta) = 2a$

$\Rightarrow \quad \log r + \log(1-\cos\theta) = \log 2a$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0 \text{ or } \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\therefore \quad \cot\phi = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -\cot(\theta/2)$$

ie., $\quad \cot\phi = \cot(-\theta/2) \Rightarrow \phi = -(\theta/2)$

Consider $p = r\sin\phi$

$\therefore \quad p = r\sin(-\theta/2)$ or $p = -r\sin(\theta/2)$

Now we have, $\quad r(1-\cos\theta) = 2a \quad \ldots(1)$

$\quad p = -r\sin(\theta/2) \quad \ldots(2)$

We have to eliminate θ from (1) and (2).

(1) can be put in the form $r \cdot 2\sin^2(\theta/2) = 2a$

ie., $\quad r\sin^2(\theta/2) = a.$

But $p/-r = \sin(\theta/2)$, from (2).

$$\therefore \quad r\left(\frac{p^2}{r^2}\right) = a \text{ or } p^2 = ar$$

Thus $p^2 = ar$ is the required pedal equation.

74. $r^2 = a^2\sec 2\theta$

$\Rightarrow \quad 2\log r = 2\log a + \log(\sec 2\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{2\sec 2\theta\tan 2\theta}{\sec 2\theta} \text{ ie., } \frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$$

ie., $\quad \cot\phi = \cot(\pi/2 - 2\theta) \Rightarrow \phi = \pi/2 - 2\theta$

Consider $p = r\sin\phi \;\therefore\; p = r\sin(\pi/2 - 2\theta)$ *ie.,* $p = r\cos 2\theta$

Now we have, $r^2 = a^2 \sec 2\theta$ ... (1)

$p = r \cos 2\theta$ ... (2)

From (2), $p/r = \cos 2\theta$ or $r/p = \sec 2\theta$

Substituting in (1) we get, $r^2 = a^2 (r/p)$ or $pr = a^2$

Thus $pr = a^2$ is the required pedal equation.

---

75. $r^n = a^n \cos n\theta$

$\Rightarrow$ $n \log r = n \log a + \log(\cos n\theta)$.

Differentiating *w.r.t* θ, we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta} \quad ie., \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$\therefore$ $\cot\phi = \cot(\pi/2 + n\theta) \Rightarrow \phi = \pi/2 + n\theta$

Consider $p = r \sin\phi$

$\therefore$ $p = r \sin(\pi/2 + n\theta)$ *ie.,* $p = r \cos n\theta$

Now we have, $r^n = a^n \cos n\theta$ ... (1)

$p = r \cos n\theta$ ... (2)

$\therefore$ (1) as a consequence of (2) is $r^n = a^n (p/r)$

Thus $r^{n+1} = p a^n$ is the required pedal equation.

---

76. $r^m = a^m (\cos m\theta + \sin m\theta)$

$\Rightarrow$ $m \log r = m \log a + \log(\cos m\theta + \sin m\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{-m \sin m\theta + m \cos m\theta}{\cos m\theta + \sin m\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} = \frac{\cos m\theta(1 - \tan m\theta)}{\cos m\theta(1 + \tan m\theta)}$$

$\therefore$ $\cot\phi = \cot(\pi/4 + m\theta) \Rightarrow \phi = \pi/4 + m\theta$

Consider $p = r \sin\phi$

$\therefore$ $p = r \sin(\pi/4 + m\theta)$

ie., $p = r[\sin(\pi/4)\cos m\theta + \cos(\pi/4)\sin m\theta]$

ie., $p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$

(We have used the formula of $\sin(A+B)$ *and also the values* $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$)

Now we have, $r^m = a^m(\cos m\theta + \sin m\theta)$ ... (1)

$$p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta) \quad \text{... (2)}$$

Using (2) in (1) we get,

$$r^m = a^m \cdot \frac{p\sqrt{2}}{r} \quad \text{or} \quad r^{m+1} = \sqrt{2}\, a^m p$$

Thus $r^{m+1} = \sqrt{2}\, a^m p$ **is the required pedal equation.**

---

77. $r = 2(1+\cos\theta)$

$\Rightarrow$ $\log r = \log 2 + \log(1+\cos\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

$\therefore$ $\cot\phi = \cot(\pi/2+\theta/2) \Rightarrow \phi = \pi/2+\theta/2$

Consider $p = r\sin\phi$

$\therefore$ $p = r\sin(\pi/2+\theta/2) = r\cos(\theta/2)$

Now we have, $r = 2(1+\cos\theta)$ ... (1)

$$p = r\cos(\theta/2) \quad \text{... (2)}$$

(1) can be put in the form $r = 2 \cdot 2\cos^2(\theta/2)$

ie., $r = 4\cos^2(\theta/2)$

From (2), $p/r = \cos(\theta/2)$ and hence (1) becomes,

$$r = 4 \cdot (p^2/r^2) \quad \text{or} \quad r^3 = 4p^2$$

Thus $r^3 = 4p^2$ is the required pedal equation.

---

78. $l/r = 1 + e\cos\theta$

$\Rightarrow$ $\log l - \log r = \log(1+e\cos\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$$

ie., $\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$ We cannot find ϕ explicitly.

Consider $p = r\sin\phi$

By squaring and taking the reciprocal we have,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

Substituting for $\cot\phi$ itself we have

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right\} \quad \ldots (1)$$

Also we have $\frac{l}{r} = 1+e\cos\theta$ ... (2)

We need to eliminate θ from (1) and (2).

From (2) $\frac{l}{r}-1 = e\cos\theta$ ... (3)

Also $e^2\sin^2\theta = e^2(1-\cos^2\theta) = e^2-e^2\cos^2\theta$

By using (3) we have $e^2\sin^2\theta = e^2-\left(\frac{l}{r}-1\right)^2$ ... (4)

Now substituting (3) and (4) in (1) we have,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2-\left(\frac{l}{r}-1\right)^2}{(l^2/r^2)}\right\}$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left\{e^2-\left(\frac{l}{r}-1\right)^2\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left\{e^2-\frac{l^2}{r^2}+\frac{2l}{r}-1\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}+\frac{e^2}{l^2}-\frac{1}{r^2}+\frac{2}{lr}-\frac{1}{l^2}$$

Thus $\frac{1}{p^2} = \frac{e^2-1}{l^2}+\frac{2}{lr}$ is the required pedal equation.

79. $r^n = a^n \operatorname{sech} n\theta$

$\Rightarrow \quad n \log r = n \log a + \log(\operatorname{sech} n\theta)$

Differentiating $w.r.t\ \theta$, we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \operatorname{sech} n\theta \tanh n\theta}{\operatorname{sech} n\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = -n \tanh n\theta$$

$\therefore \quad \cot\phi = -\tanh n\theta$ and ϕ cannot be found explicitly.

Consider $p = r\sin\phi$. Squaring and taking the reciprocal, we get

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\tanh^2 n\theta) \qquad \ldots (1)$$

Also we have, $r^n = a^n \operatorname{sech} n\theta$

$$\therefore \quad \frac{r^n}{a^n} = \operatorname{sech} n\theta \quad \text{and we have} \quad 1-\tanh^2 n\theta = \operatorname{sech}^2 n\theta$$

$$\therefore \quad \tanh^2 n\theta = 1-\operatorname{sech}^2 n\theta = 1-\left(\frac{r^n}{a^n}\right)^2$$

Substituting this expression in the R.H.S of (1) we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{2-\frac{r^{2n}}{a^{2n}}\right\}$$ being the required pedal equation

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80. [illegible] a and α are constant [illegible] that the tangent [illegible] the radius [illegible] find the pedal equation of the [illegible]

>> We have $r = a\,e^{\theta\cot\alpha}$

$\Rightarrow \quad \log r = \log a + \theta\cot\alpha\log e \quad$ But $\log e = 1$

$\therefore \quad \log r = \log a + \cot\alpha\cdot\theta$

Differentiating $w.r.t\ \theta$, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha\cdot 1$$

ie., $\cot\phi = \cot\alpha \Rightarrow \phi = \alpha = \textbf{constant}$

$\therefore$ **the tangent is inclined at a constant angle with the radius vector.**

Consider $p = r\sin\phi$. But $\phi = \alpha$

$\therefore \quad p = r\sin\alpha$. This is independent of θ.

Hence $p = r\sin\alpha$ is the required pedal equation.

81. *Show that for the curve* $r\cos\left(\sqrt{a^2-b^2}/a\right)\theta = \sqrt{a^2-b^2}$, $p^2(r^2+b^2) = a^2r^2$

>> We have $r\cos\left(\sqrt{a^2-b^2}/a\right)\theta = \sqrt{a^2-b^2}$

For convenience let $\sqrt{a^2-b^2}/a = k$, a constant.

We now have $r\cos k\theta = ka$

$\Rightarrow \quad \log r + \log(\cos k\theta) = \log(ka)$.

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{-k\sin k\theta}{\cos k\theta} = 0$$

ie., $\cot\phi = k\tan k\theta$. We cannot find ϕ explicitly.

Consider $p = r\sin\phi$.

Squaring and taking the reciprocal, we have

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \qquad \text{or} \qquad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}(1+k^2\tan^2 k\theta) \qquad \ldots (1)$$

But $r\cos k\theta = ka \qquad \ldots (2)$

We need to eliminate θ from (1) and (2).

From (2), $\cos k\theta = \dfrac{ka}{r} \Rightarrow \sec k\theta = \dfrac{r}{ka}$

Now $\tan^2 k\theta = \sec^2 k\theta - 1 = \dfrac{r^2}{k^2a^2} - 1$

Substituting this expression in (1) we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1 + k^2\left(\frac{r^2}{k^2a^2} - 1\right)\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{r^2}{a^2} - k^2\right\} \quad \text{But } k^2 = \frac{a^2 - b^2}{a^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{r^2}{a^2} - \frac{a^2 - b^2}{a^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{a^2 + r^2 - a^2 + b^2}{a^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{r^2 + b^2}{r^2 a^2}$$

Thus $p^2(r^2 + b^2) = a^2 r^2$ as required.

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82. [illegible]

Note : *Observing the complexity of the given equation we do not venture to take logarithms.*

>> We have $a\,\theta = \sqrt{r^2 - a^2} - a\cos^{-1}(a/r)$

Differentiating *w.r.t.* θ on both sides keeping in mind that r is a function of θ we obtain

$$a = \frac{1}{2\sqrt{r^2 - a^2}} \cdot 2r\frac{dr}{d\theta} - a \cdot -\frac{1}{\sqrt{1 - (a^2/r^2)}} \cdot -\frac{a}{r^2}\frac{dr}{d\theta}$$

ie., $$a = \frac{r}{\sqrt{r^2 - a^2}} \cdot \frac{dr}{d\theta} - \frac{a^2}{r^2}\frac{r}{\sqrt{r^2 - a^2}}\frac{dr}{d\theta}$$

ie., $$a = \frac{r}{\sqrt{r^2 - a^2}} \cdot \frac{dr}{d\theta}\left(1 - \frac{a^2}{r^2}\right)$$

ie., $$a = \frac{r}{\sqrt{r^2 - a^2}} \cdot \frac{dr}{d\theta}\left(\frac{r^2 - a^2}{r^2}\right)$$

ie., $$a = \frac{\sqrt{r^2 - a^2}}{r}\frac{dr}{d\theta} \quad \text{or} \quad r\frac{d\theta}{dr} = \frac{\sqrt{r^2 - a^2}}{a}$$

ie., $$\tan\phi = \frac{\sqrt{r^2 - a^2}}{a} \quad \therefore \quad \phi = \tan^{-1}\left(\frac{\sqrt{r^2 - a^2}}{a}\right)$$

---

83. *Establish the pedal equation of the curve*
$r^n = a^n \sin n\theta + b^n \cos n\theta$ *in the form* $p^2(a^{2n} + b^{2n}) = $ [illegible]

>> We have $r^n = a^n \sin n\theta + b^n \cos n\theta$

$\Rightarrow$ $n\log r = \log(a^n \sin n\theta + b^n \cos n\theta)$

Differentiating *w.r.t* θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{na^n \cos n\theta - nb^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Dividing by n, $\cot\phi = \dfrac{a^n \cos n\theta - b^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$

Consider $p = r\sin\phi$

Since ϕ cannot be found, squaring and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{(a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{(a^n \sin n\theta + b^n \cos n\theta)^2 + (a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{a^{2n}(\sin^2 n\theta + \cos^2 n\theta) + b^{2n}(\cos^2 n\theta + \sin^2 n\theta)}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

(Product terms cancels out in the numerator)

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{a^{2n}+b^{2n}}{(a^n \sin n\theta + b^n \cos n\theta)^2}$$

or $$\frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{a^{2n}+b^{2n}}{(r^n)^2}$$, by using the given equation

Thus $p^2(a^{2n}+b^{2n}) = r^{2n+2}$ is the required pedal equation.

84. *Find the length of the perpendicular from the pole to the tangent at the point $\theta = \pi/2$ on the curve $r = a(1-\cos\theta)$*

>> We have $r = a(1-\cos\theta)$

$\Rightarrow$ $\log r = \log a + \log(1-\cos\theta)$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$$

ie., $\cot\phi = \cot(\theta/2) \Rightarrow \phi = \theta/2$

Length of the perpendicular $p = r\sin\phi$

ie., $p = r\sin(\theta/2)$

Substituting $(r, \theta) = (a, \pi/2)$ we get $p = a\sin(\pi/4)$

Thus $p = a/\sqrt{2}$

85. *Find the length of the perpendicular from the pole to the tangent at the point $\theta = \pi/3$ on the curve $r = a\sec^2(\theta/2)$*

>> We have $r = a\sec^2(\theta/2)$

$\Rightarrow \quad \log r = \log a + 2\log\sec(\theta/2)$

Differentiating *w.r.t* θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2\cdot\sec(\theta/2)\tan(\theta/2)\cdot 1/2}{\sec(\theta/2)} = \tan(\theta/2)$$

$\therefore \quad \cot\phi = \cot(\pi/2 - \theta/2) \Rightarrow \phi = \pi/2 - \theta/2$

Length of the perpendicular $p = r\sin\phi$

ie., $p = r\sin(\pi/2 - \theta/2)$ or $p = r\cos(\theta/2)$

We have at $\theta = \pi/3$, $r = a\sec^2(\pi/6) = 4a/3$

$$\therefore \quad p = \frac{4a}{3}\cos(\pi/6) = \frac{4a}{3}\cdot\frac{\sqrt{3}}{2} = \frac{2a}{\sqrt{3}}$$

Hence the length of the perpendicular $p = 2a/\sqrt{3}$

EXERCISES

Find the angle between the radius vector and the tangent for the following curves.

1. $r\sec^2(\theta/2) = 2a$
2. $r = a\,\mathrm{cosec}^2(\theta/2)$
3. $r^2 = a^2(\cos 2\theta + \sin 2\theta)$
4. $r^n\,\mathrm{cosec}\,n\theta = a^n$

Find the slopes of the tangents for the following curves at the indicated points.

5. $r^2 = a^2\sin 2\theta$ at $\theta = \pi/12$
6. $r\,\mathrm{cosec}\,2\theta = a$ at $\theta = \pi/4$
7. $r = a\sin 3\theta$ at the pole
8. $r\sec^2(\theta/2) = 4$ at $\theta = \pi/2$

Show that the following pairs of curves intersect each other orthogonally.

9. $r\sec^2(\theta/2) = a$ and $r\,\mathrm{cosec}^2(\theta/2) = b$
10. $r^n\cos n\theta = a^n$ and $r^n\sin n\theta = b^n$

11. $2a/r = 1 + \cos\theta$ and $2a/r = 1 - \cos\theta$

12. $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$

Find the angle of intersection for the following pairs of curves.

13. $r = a\cos\theta$ and $r = a/2$

14. $r^n = a^n(\sin n\theta + \cos n\theta)$ and $r^n = a^n \sin n\theta$

15. $r^2 \cos(2\theta + \alpha) = a^2$ and $r^2 \cos(2\theta + \beta) = b^2$

16. $r^2 = a^2 \cos 2\theta + b^2$ and $r = b$

Obtain the pedal equation of the following curves.

17. $r^2 \cos 2\theta = a^2$

18. $r = 2a/1 + \cos\theta$

19. $r = a\,\mathrm{sech}\, n\theta$

20. $r = a + b\cos\theta$

21. $r^2 = a^2 \sin 2\theta + b^2 \cos 2\theta$

22. $r = a\sin 3\theta$

23. $r^n \sec n\theta = a^n$

24. Show that for the curve $r \sin^2(\theta/2) = a$ the length of the perpendicular from the pole to the tangent at the point $(2a\ \pi/2)$ on the curve is equal to $a\sqrt{2}$.

25 Show that the length of the perpendicular from the pole to the tangent at the point $\theta = \pi/6$ on the curve $r^2 \cos 2\theta = a^2$ is equal to $a/\sqrt{2}$.

ANSWERS

1. $\pi/2 + \theta/2$
2. $-\theta/2$
3. $\pi/4 + 2\theta$
4. $n\theta$
5. 1
6. -1
7. 0
8. 1

13. $\pi/3$
14. $\pi/4$
15. $\alpha - \beta$
16. $\tan^{-1}(a^2/b^2)$
17. $p\,r = a^2$
18. $p^2 = ar$
19. $\dfrac{1}{p^2} = \dfrac{n^2+1}{r^2} - \dfrac{n^2}{a^2}$
20. $p^2[2ar + b^2 - a^2] = r^4$
21. $r^6 = p^2(a^4 + b^4)$
22. $p^2(9a^2 - 8r^2) = r^4$
23. $p\,a^n = r^{n+1}$

2.3 Derivative of Arc length

2.31 Introduction

An arc of a curve is a part of it and we are familiar with the various form of curves. Cartesian form [$y = f(x)$], Parametric form [$x = x(t),\ y = y(t)$], Polar form [$r = f(\theta)$], pedal form [$f(r, p) = c$] Length of an arc of a curve is usually denoted by 's' and several results connected with the derivative of arc length 's' can be established from the basic definition. **We assume** these well established results as it is an **essential pre-requisite** for the study of the following topic called **Radius of Curvature**

2.32 Formulae connected with the derivative of arc length

1. **Cartesian curve :** $y = f(x)$

(i) $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ (ii) $\sin\psi = \dfrac{dy}{ds}$ (iii) $\cos\psi = \dfrac{dx}{ds}$ (iv) $\tan\psi = \dfrac{dy}{dx}$

ψ being the angle made by the tangent at $P(x, y)$ on the curve with the X – axis

2. **Parametric curve :** $x = x(t), y = y(t)$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

3. **Polar curve :** $r = f(\theta)$

(i) $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$ (ii) $\sin\phi = r\,\dfrac{d\theta}{ds}$ (iii) $\cos\phi = \dfrac{dr}{ds}$

ϕ being the angle made by the radius vector and the tangent at $P(r, \theta)$ on the polar curve $r = f(\theta)$.

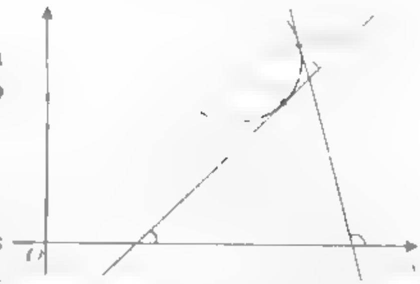
2.4 Radius of curvature

2.41 Introduction

If we traverse in a ghat section (hilly region) where the road is not straight, we often see caution boards "sharp bend ahead", "hairpin bend ahead" etc which gives an indication of the difference in the amount of bending of a road at various points which is nothing but *curvature* at various points and we discuss the same in a mathematical way. This aspect is discussed for cartesian, parametric, polar and pedal form of curves

2.42 Curvature and Radius of Curvature

Definition



Consider a curve in the XOY plane and let A be a fixed point on it. Let P and Q be two neighbouring points on the curve such that,

$\overset{\frown}{AP} = s$ and $\overset{\frown}{AQ} = s + \delta s$ so that $\overset{\frown}{PQ} = \delta s$.

Let ψ and $\psi + \delta\psi$ respectively be the angles made by the tangents at P and Q with the X-axis.

The angle $\delta\psi$ between the tangents is called the bending of the curve which depends on δs. $\delta\psi/\delta s$ is called as the *mean curvature* of the arc PQ. Also the amount of bending of the curve at P is called as the ***curvature*** of the curve at P and is defined mathematically as

$$\lim_{\substack{\delta s \to 0 \\ (Q \to P)}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds} \text{ be denoted by } K$$

ie., ***Curvature*** $= K = \dfrac{d\psi}{ds}$. Further if $K \neq 0$, the reciprocal of the curvature is called as the *radius of curvature* and is denoted by ρ.

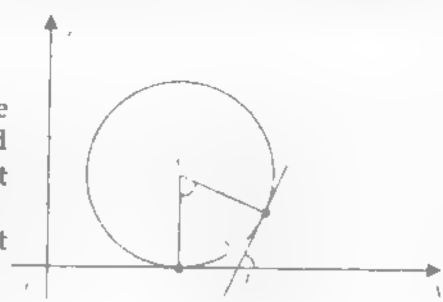
ie., ***Radius of curvature*** $= \rho = \dfrac{1}{K} = \dfrac{ds}{d\psi}$

Note :

1. *As it is obvious that ψ depends on s, the relationship between these is called as the intrinsic equation and (s, ψ) are called the intrinsic coordinates of the point P*
2. *We always take the sign of K and ρ to be positive.*

Remark : *Curvature being the amount of bending is obviously zero for a straight line at all the points on it. It is easy to visualize that the circle has an uniform bending and hence the curvature of a circle is a constant which will be established mathematically.*

A Question Format : *Define curvature and prove that the curvature of a circle is a constant.*



[Definition given already]

Consider a circle of radius r having centre at the point C. Let A be a fixed point on the circle and $P(x, y)$ be any point on the circle such that $\overset{\frown}{AP} = s$. Let ψ be the angle made by the tangent

at P with the X-axis at the point B (*interior angle being* $\pi - \psi$) Clearly $CA = CP = r =$ radius.

We have from the quadrilateral $CABP$, $\hat{C} + \hat{A} + \hat{B} + \hat{P} = 2\pi$

ie., $\hat{C} + \pi/2 + (\pi - \psi) + \pi/2 = 2\pi \quad \therefore \quad A\hat{C}P = \psi$

We have a known result

$$s = r\psi \quad \text{or} \quad \psi = \frac{s}{r} \quad \therefore \quad \frac{d\psi}{ds} = \frac{1}{r} = \text{constant.}$$

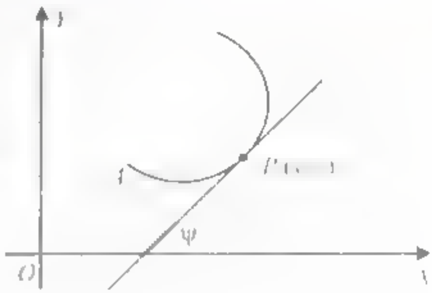
Thus the curvature $K = 1/r =$ **constant.**

This proves that the curvature of a circle at any point on it is a constant and is equal to the reciprocal of the radius.

We now proceed to derive expressions for the radius of curvature in respect of cartesian, parametric, polar and pedal form of curves.

2.43 An expression for the radius of curvature in the case of a cartesian curve

Let $y = f(x)$ be the equation of the cartesian curve and A be a fixed point on it. Let $P(x, y)$ be a point on the curve such that $\overset{\frown}{AP} = s$. Let ψ be the angle made by the tangent at P with the x-axis.



Then we know that $\tan\psi = \dfrac{dy}{dx}$

Differentiating w.r.t s we have,

$$\frac{d}{ds}(\tan\psi) = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

ie., $$\sec^2\psi \, \frac{d\psi}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds}$$

But $\dfrac{dx}{ds} = \cos\psi$ and by the definition $\dfrac{d\psi}{ds} = \dfrac{1}{\rho}$

$$\therefore \quad \sec^2\psi \cdot \frac{1}{\rho} = \frac{d^2y}{dx^2}\cos\psi \quad \text{or} \quad \sec^3\psi = \rho\,\frac{d^2y}{dx^2}$$

Hence $\rho = \sec^3\psi \Big/ \dfrac{d^2y}{dx^2}$

ie.,
$$\rho = \frac{(\sec^2\psi)^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}$$

Denoting $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2}$ we have,

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

Note : *Sometimes* y_1 *at some point on the curve becomes infinity (ie., when the tangent is perpendicular to the x-axis,* $\tan\psi = \tan 90° = \infty$*) in which case we cannot apply the formula for* ρ *in the above form. In such a case we have to use the formula in the alternative form,*

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \quad \text{where } x_1 = \frac{dx}{dy} \text{ and } x_2 = \frac{d^2x}{dy^2}$$

2.44 An expression for the radius of curvature in the case of a parametric curve $x = x(t),\ y = y(t)$

[**Note** : *The expression for* ρ *in the case of* $y = f(x)$ *has to be established first.*]

We have for a cartesian curve $y = f(x)$,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad \ldots (1)$$

We shall express y_1 and y_2 in terms of the parameter t

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{y'}{x'} \quad \text{where } y' = \frac{dy}{dt},\ x' = \frac{dx}{dt}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx} = \frac{x'y''-y'x''}{(x')^2} \cdot \frac{1}{x'}$$

ie.,
$$y_2 = \frac{x'y''-y'x''}{(x')^3} \quad \text{where } y'' = \frac{d^2y}{dt^2} \text{ and } x'' = \frac{d^2x}{dt^2}$$

Substituting in (1) we get,

$$\rho = \frac{\{1+(y'/x')^2\}^{3/2}}{x'y''-y'x''}\cdot(x')^3$$

$$= \frac{\{(x')^2+(y')^2\}^{3/2}}{\{(x')^2\}^{3/2}\cdot(x'y''-y'x'')}\cdot(x')^3$$

$$= \frac{\{(x')^2+(y')^2\}^{3/2}}{(x')^3(x'y''-y'x'')}\cdot(x')^3$$

Thus $$\boxed{\rho = \frac{\{(x')^2+(y')^2\}^{3/2}}{x'y''-y'x''}}$$

In an alternative notation with the same meaning the above expression is also put in the form

$$\rho = \frac{\{(\dot{x})^2+(\dot{y})^2\}^{3/2}}{\dot{x}\,\ddot{y}-\dot{y}\,\ddot{x}}$$

Note for problems

We prefer to use the cartesian formula itself for finding ρ in the case of parametric curves also as the work will be relatively easy.

WORKED PROBLEMS

86. *Find the radius of curvature of the curve whose intrinsic equation is* $s = a\log\tan(\pi/4+\psi/2)$

\>> $s = a\log\tan(\pi/4+\psi/2)$ and we have $\rho = \dfrac{ds}{d\psi}$

Differentiating w.r.t ψ we have,

$$\frac{ds}{d\psi} = a\cdot\frac{1}{\tan(\pi/4+\psi/2)}\cdot\sec^2(\pi/4+\psi/2)\cdot\frac{1}{2}$$

$$= \frac{a}{2}\cdot\frac{\cos(\pi/4+\psi/2)}{\sin(\pi/4+\psi/2)}\cdot\frac{1}{\cos^2(\pi/4+\psi/2)}$$

$$= \frac{a}{2\sin(\pi/4+\psi/2)\cos(\pi/4+\psi/2)}$$ But $2\sin\theta\cos\theta = \sin 2\theta$

$$\therefore\quad \frac{ds}{d\psi} = \frac{a}{\sin[2(\pi/4+\psi/2)]} = \frac{a}{\sin(\pi/2+\psi)} = \frac{a}{\cos\psi} = a\sec\psi$$

Thus $\boldsymbol{\rho = a\sec\psi}$

87. *Show that the radius of curvature for the catenary of uniform strength $y = a \log \sec (x/a)$ is $a \sec (x/a)$*

>> We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Consider $y = a \log \sec (x/a)$

$$\therefore \quad \frac{dy}{dx} = y_1 = \frac{a}{\sec (x/a)} \cdot \sec (x/a) \tan (x/a) \cdot \frac{1}{a}$$

ie., $y_1 = \tan (x/a)$. Also $y_2 = \dfrac{1}{a} \sec^2 (x/a)$

Hence $\rho = \dfrac{[1+\tan^2 (x/a)]^{3/2} \cdot a}{\sec^2 (x/a)} = \dfrac{a [\sec^2 (x/a)]^{3/2}}{\sec^2 (x/a)}$

ie., $\rho = \dfrac{a \sec^3 (x/a)}{\sec^2 (x/a)} = a \sec (x/a)$

Thus $\rho = a \sec (x/a)$

---

88. *Show that for the catenary $y = c \cosh (x/c)$ the radius of curvature is equal to y^2/c which is also equal to the length of the normal intercepted between the curve and the x-axis.*

>> We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$y = c \cosh (x/c)$ by data

$$\therefore \quad y_1 = c \cdot \sinh (x/c) \cdot \frac{1}{c} = \sinh (x/c) \; ; \; y_2 = \frac{1}{c} \cosh (x/c)$$

Hence $\rho = \dfrac{[1+\sinh^2 (x/c)]^{3/2} c}{\cosh (x/c)} = \dfrac{c [\cosh^2 (x/c)]^{3/2}}{\cosh (x/c)}$

ie., $\rho = \dfrac{c \cosh^3 (x/c)}{\cosh (x/c)} = c \cosh^2 (x/c)$

But $y/c = \cosh (x/c)$ and hence $\rho = c \cdot (y^2/c^2) = y^2/c$

Also we know that the length of the normal (l) is $y \sqrt{1+y_1^2}$

$$l = c \cosh (x/c) \sqrt{1+\sinh^2 (x/c)} = c \cosh^2 (x/c) = y^2/c$$

This proves the required result

---

89. *Find the radius of curvature for the curve* $y = ax^2 + bx + c$ *at* $x = \dfrac{1}{2a}[\sqrt{a^2-1} - b]$

>> $y = ax^2 + bx + c$, by data.

$\therefore \quad y_1 = 2ax + b, \; y_2 = 2a$

At the given point, $y_1 = 2a \cdot \dfrac{1}{2a}[\sqrt{a^2-1} - b] + b = \sqrt{a^2-1}$ and $y_2 = 2a$ itself.

We have, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{[1+(a^2-1)]^{3/2}}{2a} = \frac{(a^2)^{3/2}}{2a} = \frac{a^2}{2}$$

Thus $\rho = a^2/2$

---

90. *Find the radius of curvature for the Folium of De-Cartes* $x^3 + y^3 = 3axy$ *at the point* $(3a/2, 3a/2)$ *on it.*

>> $x^3 + y^3 = 3axy$, by data.

Differentiating w.r.t. x we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

ie, $3(y^2 - ax)\dfrac{dy}{dx} = 3(ay - x^2) \quad \therefore \; \dfrac{dy}{dx} = y_1 = \dfrac{ay - x^2}{y^2 - ax}$

At $(3a/2, 3a/2)$, $y_1 = \dfrac{3a^2/2 - 9a^2/4}{9a^2/4 - 3a^2/2} = -1$

Next $$\frac{d^2y}{dx^2} = y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

At $(3a/2, 3a/2)$ we note that, $y^2 - ax = 9a^2/4 - 3a^2/2 = 3a^2/4$ and $ay - x^2 = 3a^2/2 - 9a^2/4 = -3a^2/4$.

Hence at $(3a/2, 3a/2)$,

$$y_2 = \frac{(3a^2/4)(-a-3a) - (-3a^2/4)(-3a-a)}{(3a^2/4)^2}$$

ie., $$y_2 = \frac{-3a^3 - 3a^3}{9a^4/16} = \frac{16(-6a^3)}{9a^4} = \frac{-32}{3a}$$

We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Hence $\rho = \dfrac{(1+1)^{3/2}}{-32/3a} = \dfrac{2\sqrt{2}\cdot 3a}{-32} = \dfrac{-3\sqrt{2}\,a}{16} = \dfrac{-3a}{8\sqrt{2}}$

Thus $\quad |\,\rho\,| = \mathbf{3a/8\sqrt{2}}$

---

91. *Find the radius of curvature for the curve $y^2 = \dfrac{4a^2(2a-x)}{x}$ where the curve meets the x-axis*

>> If the curve meets the x-axis then $y = 0$.

$\therefore \qquad \dfrac{4a^2(2a-x)}{x} = 0 \Rightarrow 4a^2(2a-x) = 0 \qquad \therefore\ x = 2a$

Thus $(2a, 0)$ is the point on the curve at which we have to find ρ

The given equation can be put in the form

$$y^2 = \frac{8a^3}{x} - 4a^2$$

Differentiating w.r.t x we have $2yy_1 = -\dfrac{8a^3}{x^2}$ or $y_1 = \dfrac{-4a^3}{x^2 y}$

At $(2a, 0)$ y_1 becomes infinity and hence we have to consider dx/dy.

Let $\quad x_1 = \dfrac{dx}{dy} = \dfrac{-x^2 y}{4a^3}$ and $x_1 = 0$ at $(2a, 0)$

Now $\quad x_2 = \dfrac{d^2 x}{dy^2} = \dfrac{-1}{4a^3}\,[\,x^2\cdot 1 + y\cdot 2x\,x_1\,]$

$\therefore \qquad$ at $(2a, 0)$: $x_2 = -4a^2/4a^3 = -1/a$

We have $\rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$

$$= \frac{(1+0)^{3/2}}{-1/a} = -a$$

Thus $\quad |\,\rho\,| = \mathbf{a}$

---

92. *Find the radius of curvature for the curve $x^2 y = a(x^2+y^2)$ at the point $(-2a, 2a)$*

>> Consider $x^2 y = a(x^2+y^2)$ and differentiate w.r.t. x

$\therefore \qquad x^2 y_1 + 2xy = 2ax + 2ayy_1$

ie., $\quad y_1(x^2-2ay) = 2ax-2xy$

or $\quad y_1 = \dfrac{2ax-2xy}{x^2-2ay}$; At $(-2a, 2a)$, y_1 is infinity.

Hence $x_1 = \dfrac{dx}{dy} = \dfrac{1}{y_1} = \dfrac{x^2-2ay}{2ax-2xy}$ and at $(-2a, 2a)$ we have $x_1 = 0$

Also $\quad \dfrac{d^2x}{dy^2} = \dfrac{(2ax-2xy)(2xx_1-2a)-(x^2-2ay)(2ax_1-2x-2x_1y)}{(2ax-2xy)^2}$

We note that at $(-2a, 2a)$

$$(2ax-2xy) = 4a^2 \text{ and } (x^2-2ay) = 0$$

$\therefore \quad (x_2)_{(-2a,2a)} = \dfrac{(4a^2)(-2a)}{16a^4} = \dfrac{-1}{2a}$

We have, $\quad \rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$

$$= \frac{(1)^{3/2}}{-1/2a} = -2a$$

Thus $\quad |\rho| = 2a$

---

93. Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y} = 4$ at the point where it cuts the line passing through the origin making an angle $45°$ with the x-axis.

>> The equation of the line is $y = x$ and we shall find the point of intersection of this line with the curve $\sqrt{x}+\sqrt{y} = 4$.

This equation when $y = x$ becomes,

$$\sqrt{x}+\sqrt{x} = 4 \text{ or } 2\sqrt{x} = 4 \text{ or } \sqrt{x} = 2 \text{ or } x = 4$$

$\therefore$ the point of intersection is $(4, 4)$

Consider $\sqrt{x}+\sqrt{y} = 4$ and differentiate w.r.t. x

$\therefore \quad \dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{y}}\,y_1 = 0 \quad$ or $\quad \dfrac{y_1}{\sqrt{y}} = \dfrac{-1}{\sqrt{x}}$

ie., $\quad y_1 = -\sqrt{y}/\sqrt{x}$. At $(4,4)$ we get $y_1 = -1$

Now $\quad y_2 = \dfrac{d^2y}{dx^2} = \dfrac{\sqrt{x}(-1/2\sqrt{y}\cdot y_1)-(-\sqrt{y})(1/2\sqrt{x})}{x}$

$\therefore$ at $(4, 4)$, $y_2 = \dfrac{1/2 + 1/2}{4} = \dfrac{1}{4}$

We have, $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{(2)^{3/2}}{1/4} = 4 \cdot 2\sqrt{2} = 8\sqrt{2}$$

Thus $\rho = 8\sqrt{2}$

---

94. *For the curve $y = ax/(a+x)$ show that $(2\rho/a)^{2/3} = (x/y)^2 + (y/x)^2$*

>> $y = \dfrac{ax}{a+x}$, by data.

$\therefore$ $y_1 = \dfrac{(a+x)a - ax \cdot 1}{(a+x)^2} = \dfrac{a^2}{(a+x)^2}$

Also $y_2 = \dfrac{-2a^2}{(a+x)^3}$

We have $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

Hence $\rho = \dfrac{\left[1 + \dfrac{a^4}{(a+x)^4}\right]^{3/2} \cdot (a+x)^3}{-2a^2}$

$$= \frac{[(a+x)^4 + a^4]^{3/2} \cdot (a+x)^3}{-2a^2 \{(a+x)^4\}^{3/2}}$$

$$= \frac{[(a+x)^4 + a^4]^{3/2} \cdot (a+x)^3}{-2a^2 (a+x)^6}$$

or $-2\rho = \dfrac{[(a+x)^4 + a^4]^{3/2}}{a^2 (a+x)^3}$

$\Rightarrow$ $(-2\rho)^{2/3} = \dfrac{(a+x)^4 + a^4}{a^{4/3} (a+x)^2}$; We note that $(-2)^{2/3} = 2^{2/3}$

$\therefore$ $(2\rho)^{2/3} = \dfrac{1}{a^{4/3}} \left\{ (a+x)^2 + \left(\dfrac{a^2}{a+x} \right)^2 \right\}$

But $a + x = \dfrac{ax}{y}$ by data.

$$\therefore \quad (2\rho)^{2/3} = \frac{1}{a^{4/3}}\left\{\frac{a^2x^2}{y^2} + \frac{a^2y^2}{x^2}\right\}$$

$$\text{ie,} \quad (2\rho)^{2/3} = a^{2/3}\left\{\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2\right\}$$

Thus $\quad (2\rho/a)^{2/3} = (x/y)^2 + (y/x)^2$

---

95. *Find the radius of curvature of the curve* $x = a\log(\sec t + \tan t)$, $y = a\sec t$

\>> $\quad x = a\log(\sec t + \tan t)$

$$\frac{dx}{dt} = \frac{a}{\sec t + \tan t}\cdot \sec t\tan t + \sec^2 t = \frac{a\sec t(\sec t + \tan t)}{\sec t + \tan t}$$

$$\therefore \quad \frac{dx}{dt} = a\sec t$$

Also $\quad y = a\sec t \quad$ gives $\dfrac{dy}{dt} = a\sec t\tan t$

Now, $y_1 = \dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{a\sec t\tan t}{a\sec t} = \tan t$

Differentiating w.r.t. x we get, $y_2 = \sec^2 t\,\dfrac{dt}{dx}$

$$\therefore \quad y_2 = \sec^2 t\cdot\frac{1}{a\sec t} = \frac{\sec t}{a}$$

We have $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2}a}{\sec t} = \frac{a\sec^3 t}{\sec t}$$

Thus $\quad \rho = a\sec^2 t$

---

96. *Show that the radius of curvature at any point* θ *on the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *is* $4a\cos(\theta/2)$

\>> $\quad x = a(\theta + \sin\theta) \quad ; \quad y = a(1 - \cos\theta)$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \quad ; \quad \frac{dy}{d\theta} = a\sin\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$\therefore \quad y_1 = \tan(\theta/2)$

Differentiating w.r.t. x we get,

$$y_2 = \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{d\theta}{dx}$$

$$= \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{1}{a(1+\cos\theta)} = \frac{\sec^2(\theta/2)}{4a\cos^2(\theta/2)}$$

$$\therefore \quad y_2 = \frac{1}{4a}\sec^4(\theta/2)$$

We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+\tan^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)}$$

$$= \frac{[\sec^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)} = \frac{4a\sec^3(\theta/2)}{\sec^4(\theta/2)}$$

Thus $\quad \rho = \mathbf{4a\cos(\theta/2)}$

---

97. *Find the radius of curvature of the tractrix* $x = a[\cos t + \log\tan(t/2)]$, $y = a\sin t$

>> For the given curve we have

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan(t/2)}\cdot\sec^2(t/2)\cdot\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\cos(t/2)\sin(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{-\sin^2 t + 1}{\sin t}\right] = a\cdot\frac{\cos^2 t}{\sin t}$$

ie, $\quad \dfrac{dx}{dt} = a\cos^2 t\,\operatorname{cosec} t$

Also $\quad \dfrac{dy}{dt} = a\cos t$

Now, $\quad y_1 = \dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = \dfrac{a\cos t}{a\cos^2 t\operatorname{cosec} t} = \tan t$

Hence $y_2 = \sec^2 t \dfrac{dt}{dx} = \dfrac{\sec^2 t}{a\cos^2 t\ \mathrm{cosec}\, t} = \dfrac{\sec^4 t\ \sin t}{a}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2}\cdot a}{\sec^4 t\ \sin t} = \frac{a\sec^3 t}{\sec^4 t \sin t}$$

Thus $\rho = a\cot t$

---

98. *Find the radius of curvature of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \pi/4$*

>> $x = a\cos^3\theta$; $y = a\sin^3\theta$

$\therefore$ $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$; $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

Now $y_1 = \dfrac{dy}{dx} = \dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta} = \dfrac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$

Hence $y_2 = -\sec^2\theta\ \dfrac{d\theta}{dx} = \dfrac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \dfrac{\sec^4\theta\ \mathrm{cosec}\,\theta}{3a}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2\theta)^{3/2}\cdot 3a}{\sec^4\theta\ \mathrm{cosec}\,\theta} = \frac{3a\sec^3\theta}{\sec^4\theta\ \mathrm{cosec}\,\theta} = 3a\cos\theta\ \sin\theta$$

Thus at $\theta = \pi/4$, $\rho = 3a/2$

---

99. *Show that the radius of curvature of the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ is 'at'.*

>> $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

$\dfrac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$; $\dfrac{dy}{dt} = a(\cos t + t\sin t - \cos t)$

$\therefore$ $\dfrac{dx}{dt} = at\cos t$ and $\dfrac{dy}{dt} = at\sin t$

Now, $y_1 = \dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = \dfrac{a\,t\sin t}{a\,t\cos t} = \tan t$

Hence $y_2 = \sec^2 t \dfrac{dt}{dx} = \dfrac{\sec^2 t}{at \cos t} = \dfrac{\sec^3 t}{at}$

We have, $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{(1 + \tan^2 t)^{3/2}}{\sec^3 t} \cdot at = \frac{\sec^3 t}{\sec^3 t} \cdot at$$

Thus $\rho = at$

---

100. *If ρ be the radius of curvature at any point $P(x, y)$ on the parabola $y^2 = 4ax$ show that ρ^2 varies as $(SP)^3$ where S is the focus of the parabola.*

>> Consider $y^2 = 4ax$ and differentiate w.r.t. x

$\therefore \quad 2yy_1 = 4a$ or $y_1 = 2a/y$

Further $y_2 = \dfrac{-2a}{y^2} \cdot y_1 = \dfrac{-4a^2}{y^3}$

We have, $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{\{1 + (4a^2/y^2)\}^{3/2}}{-4a^2/y^3} = \frac{y^3 \{(y^2 + 4a^2)/y^2\}^{3/2}}{-4a^2}$$

$$= \frac{y^3}{-4a^2} \cdot \frac{(y^2 + 4a^2)^{3/2}}{(y^2)^{3/2}} = \frac{(y^2 + 4a^2)^{3/2}}{-4a^2}$$

ie., $\quad \rho = \dfrac{(4ax + 4a^2)^{3/2}}{-4a^2} = \dfrac{(4a)^{3/2}(x + a)^{3/2}}{-4a^2}$

By squaring we have,

$$\rho^2 = \frac{(4a)^3 (x + a)^3}{16a^4} = \frac{64a^3 (x + a)^3}{16a^4}$$

ie., $\quad \rho^2 = \dfrac{4}{a}(x + a)^3 \qquad \ldots (1)$

The co-ordinates of the focus of the parabola is $S = (a, 0)$ and we have $P = (x, y)$

$\therefore \quad SP = \sqrt{(x - a)^2 + (y - 0)^2}$ by the distance formula.

$$= \sqrt{x^2 - 2ax + a^2 + y^2} = \sqrt{x^2 - 2ax + a^2 + 4ax}$$

$$= \sqrt{x^2 + 2ax + a^2} = \sqrt{(x+a)^2} = (x+a)$$

Hence $SP = (x+a)$ and using this result in (1) we have,

$$\rho^2 = \frac{4}{a}(SP)^3$$

That is, $\rho^2 = \text{const.}\,(SP)^3$

Thus $\rho^2 \propto (SP)^3$

---

101. *Prove that for the ellipse $x^2/a^2 + y^2/b^2 = 1$, the radius of curvature is equal to $a^2 b^2/p^3$ where p is the length of the perpendicular from the centre of the ellipse upon the tangent at (x, y). Hence deduce that ρ at the end of the major axis is equal to the semi latus rectum*

>> The parametric equations of the ellipse are $x = a\cos\theta$, $y = b\sin\theta$ and we prefer to apply the parametric formula for finding ρ.

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \quad \text{for a parametric curve.}$$

$$\dot{x} = \frac{dx}{d\theta} = -a\sin\theta \quad ; \quad \dot{y} = \frac{dy}{d\theta} = b\cos\theta$$

$$\ddot{x} = \frac{d^2x}{d\theta^2} = -a\cos\theta \quad ; \quad \ddot{y} = \frac{d^2y}{d\theta^2} = -b\sin\theta$$

$$\therefore \quad \rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab(\sin^2\theta + \cos^2\theta)}$$

$$ie., \quad \rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

Further, equation of the tangent to the ellipse at $P\,(a\cos\theta, b\sin\theta)$ is given by $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ Also the length of the perpendicular from a point (x_1, y_1) upon a straight line $Ax + By + C = 0$ is given by the formula

$$p = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Hence the length of the perpendicular from the centre $O = (0, 0)$ of the ellipse upon the tangent $(\cos\theta/a)x + (\sin\theta/b)y - 1 = 0$ is given by

$$p = \frac{|0+0-1|}{\sqrt{\cos^2\theta/a^2 + \sin^2\theta/b^2}} = \frac{1}{\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)/a^2b^2}}$$

ie., $\quad p = \dfrac{ab}{\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}} \quad \Rightarrow p^3 = \dfrac{a^3 b^3}{(a^2 \sin^2\theta + b^2\cos^2\theta)^{3/2}}$

$\therefore \quad (a^2 \sin^2\theta + b^2 \cos^2\theta)^{3/2} = a^3 b^3/p^3 \qquad \ldots (2)$

Using (2) in (1) we get, $\rho = \dfrac{a^3 b^3/p^3}{ab} = \dfrac{a^2 b^2}{p^3}$

Thus $\rho = a^2 b^2 / p^3$

Further, at the end of the major axis we have $(x, y) = (\pm a, 0)$

$\therefore \quad a\cos\theta = \pm a \Rightarrow \cos\theta = \pm 1$ or $\cos^2\theta = 1$. $\therefore \sin^2\theta = 0$

Hence $\quad p = \dfrac{ab}{\sqrt{0 + b^2}} \quad$ or $\quad p = a$

Thus $\rho = a^2 b^2 / a^3 = b^2/a$ being the length of the semi latus rectum.

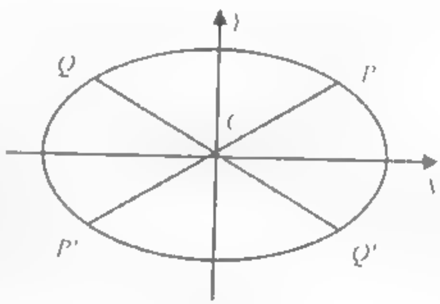
This proves both the desired results.

---

102. *If ρ_1 and ρ_2 be the radii of curvatures at the extrimities of the two conjugate diameters of the ellipse, show that*

$$\rho_1^{2/3} + \rho_2^{2/3} = (a^2 + b^2)/(ab)^{2/3}$$

>> Let PCP' and QCQ' be the two conjugate diameters of the ellipse.



Noting that $x = a\cos\theta$ and $y = b\sin\theta$ represents the parametric equations of the ellipse and recollecting a property of the canjugate diameters with reference to the ecentric angle θ, we can write

$P = (a\cos\theta,\ b\sin\theta)$ and $Q = [a\cos(\pi/2 + \theta),\ b\sin(\pi/2 + \theta)]$

Let ρ_1 be the radius of curvature at P and ρ_2 be the radius of curvature at Q.

Hence $\rho_1 = \dfrac{(a^2 \sin^2\theta + b^2 \cos^2\theta)^{3/2}}{ab} \qquad \ldots (1)$

[Refer the previous example]

Changing θ to $\pi/2 + \theta$ we have $\sin(\pi/2 + \theta) = \cos\theta$ and $\cos(\pi/2 + \theta) = -\sin\theta$. Thus we have from (1)

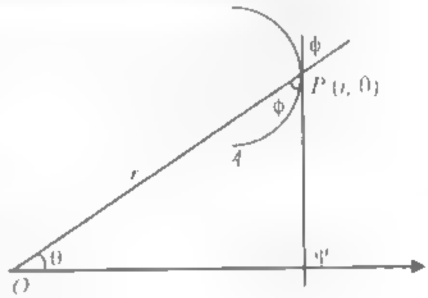
$$\rho_2 = \frac{(a^2 \cos^2\theta + b^2 \sin^2\theta)^{3/2}}{ab}$$

Hence $\rho_1^{2/3} = \dfrac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}}$; $\rho_2^{2/3} = \dfrac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}}$

$$\therefore \qquad \rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2(\sin^2\theta + \cos^2\theta) + b^2(\cos^2\theta + \sin^2\theta)}{(ab)^{2/3}}$$

Thus $\quad \boldsymbol{\rho_1^{2/3} + \rho_2^{2/3} = a^2 + b^2/(ab)^{2/3}}$

$\boxed{2.45}$ An expression for the radius of curvature in the case of a polar curve $r = f(\theta)$



Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at $P(r,\theta)$.

Let ψ be the angle made by the tangent at P with the initial line.
Let A be a fixed point on the curve and let $\overset{\frown}{AP} = s$.

We have $\psi = \theta + \phi$

$$\therefore \qquad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds} \qquad ie,\ \frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\text{or} \qquad \rho = \frac{\left(\dfrac{ds}{d\theta}\right)}{1 + \dfrac{d\phi}{d\theta}} \tag{1}$$

We know that $\tan\phi = r\,\dfrac{d\theta}{dr} = r\Big/\left(\dfrac{dr}{d\theta}\right)$

ie., $\qquad \tan\phi = \dfrac{r}{r_1}$ where $r_1 = \dfrac{dr}{d\theta}$

Differentiating w.r.t θ we get,

$$\sec^2\phi\,\frac{d\phi}{d\theta} = \frac{r_1\cdot r_1 - r\cdot r_2}{r_1^2} \quad \text{where } r_2 = \frac{d^2 r}{d\theta^2}$$

$$\text{or} \qquad \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2\sec^2\phi} = \frac{r_1^2 - r r_2}{r_1^2(1 + \tan^2\phi)}$$

ie., $$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2\left[1 + (r^2/r_1^2)\right]} = \frac{r_1^2 - r r_2}{r_1^2 + r^2}$$

Hence $$1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2 - r r_2}{r^2 + r_1^2} = \frac{r^2 + r_1^2 + r_1^2 - r r_2}{r^2 + r_1^2}$$

ie., $$1 + \frac{d\phi}{d\theta} = \frac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2} \qquad \ldots (2)$$

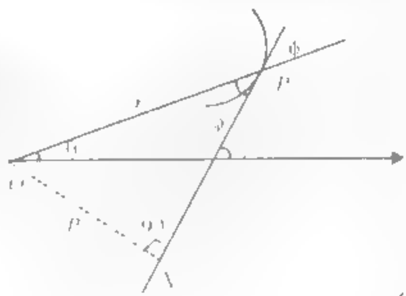
Also, we know that $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r_1^2} \qquad (3)$$

Using (2) and (3) in (1) we get

$$\rho = \sqrt{r^2 + r_1^2}\,\frac{(r^2 + r_1^2)}{r^2 + 2r_1^2 - r r_2}$$

Thus in the polar form, $$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}}$$

2.46 An expression for the radius of curvature in the case of a pedal curve



Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at P. Let ψ be the angle made by the tangent at P with the initial line. Draw $ON = p$, a perpendicular from the pole to the tangent.

We have from the Δ ONP, $\sin\phi = \dfrac{p}{r}$

ie., $$p = r \sin\phi \qquad (1)$$

Differentiating (1) w.r.t r we get,

$$\frac{dp}{dr} = r\cos\phi\,\frac{d\phi}{dr} + 1\cdot\sin\phi$$

But we know that, $\sin\phi = r\dfrac{d\theta}{ds}$ and $\cos\phi = \dfrac{dr}{ds}$

$$\therefore \quad \frac{dp}{dr} = r\,\frac{d\phi}{dr}\,\frac{dr}{ds} + r\,\frac{d\theta}{ds} = r\left[\frac{d\phi}{ds} + \frac{d\theta}{ds}\right] = r\,\frac{d}{ds}(\phi + \theta)$$

But $\phi + \theta = \psi$

$$\therefore \quad \frac{dp}{dr} = r\,\frac{d\psi}{ds} \quad \text{or} \quad \frac{ds}{d\psi} = r\,\frac{dr}{dp}$$

Thus $\boxed{\rho = r\,\dfrac{dr}{dp}}$

Note for problems

To find ρ for a polar curve $r = f(\theta)$, we have two options

(i) Applying the polar form of ρ by finding r_1 and r_2

(ii) Applying the pedal form of ρ by first finding the pedal equation of the curve as discussed already

In the case of polar curves we prefer to take logarithms first and then differentiate w.r.t. θ

WORKED PROBLEMS

103. *Show that for the equiangular spiral $r = ae^{\theta \cot \alpha}$ where a and α are constants ρ/r is a constant.*

>> $\quad r = ae^{\theta \cot \alpha}$

$\Rightarrow \quad \log r = \log a + \theta \cot \alpha \, \log e$. But $\log_e e = 1$

Differentiating w.r.t. θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1 \cdot \cot \alpha \quad ie., \quad \frac{dr}{d\theta} = r_1 = r \cot \alpha$$

Hence $\dfrac{d^2 r}{d\theta^2} = r_2 = r_1 \cot \alpha = (r \cot \alpha) \cot \alpha = r \cot^2 \alpha$

We have, $\quad \rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$$\rho = \frac{(r^2 + r^2 \cot^2\alpha)^{3/2}}{r^2 + 2r^2 \cot^2\alpha - r^2 \cot^2 \alpha} = \frac{(r^2)^{3/2}(\operatorname{cosec}^2 \alpha)^{3/2}}{r^2 (1 + \cot^2 \alpha)}$$

ie., $\quad \rho = \dfrac{r^3 \operatorname{cosec}^3 \alpha}{r^2 \operatorname{cosec}^2 \alpha} = r \operatorname{cosec} \alpha$

Thus ρ/r **= cosec** α **= constant.**

Aliter : ***By applying the pedal form of*** ρ

The pedal equation of the given curve is $p = r \sin \alpha$ *[Refer Example-80]*

Differentiating w.r.t p we get,

$$1 = \frac{dr}{dp}\sin\alpha \qquad \therefore \quad \frac{dr}{dp} = \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha$$

Hence $\rho = r\dfrac{dr}{dp} = r\operatorname{cosec}\alpha$

Thus ρ/r **= cosec** α **= constant.**

---

104. *Show that the radius of curvature of the curve* $r^n = a^n\cos n\theta$ *varies inversely as* r^{n-1}

\>> $\quad r^n = a^n\cos n\theta$

$\Rightarrow \quad n\log r = n\log a + \log(\cos n\theta)$

Differentiating w.r.t. θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n\sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$$r_1 = -r\tan n\theta$$

Hence $r_2 = \dfrac{d^2r}{d\theta^2} = -r_1\tan n\theta - n\,r\sec^2 n\theta$

We have $\quad \rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$$\therefore \quad \rho = \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r(-r_1\tan n\theta - nr\sec^2 n\theta)}$$

$$= \frac{(r^2)^{3/2}(\sec^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2(1 + \tan^2 n\theta + n\sec^2 n\theta)}$$

$$= \frac{r\sec^3 n\theta}{\sec^2 n\theta(1+n)} = \frac{r\sec n\theta}{(1+n)}$$

Thus $\quad \rho = \dfrac{r}{1+n}\sec n\theta$

But $\quad a^n/r^n = \sec n\theta$ by data .

$$\rho = \frac{r}{1+n} \cdot \frac{a^n}{r^n} = \left[\frac{a^n}{1+n} \right] \frac{1}{r^{n-1}}$$

ie., $$\rho = \text{const} \cdot \frac{1}{r^{n-1}}$$

Thus $\rho \propto 1 / r^{n-1}$

Aliter : ***By the pedal form of*** ρ

The pedal equation of the given curve is $p\,a^n = r^{n+1}$ [*Refer Example 75*]

Differentiating w.r.t. p, we get,

$$a^n = (n+1)\, r^n \frac{dr}{dp} \quad \therefore \quad \frac{dr}{dp} = \frac{a^n}{(n+1)\, r^n}$$

Hence $\rho = r\,\dfrac{dr}{dp} = r \cdot \dfrac{a^n}{(n+1)\, r^n} = \dfrac{a^n}{(n+1)} \cdot \dfrac{1}{r^{n-1}} = \text{const.}\ \dfrac{1}{r^{n-1}}$

Thus $\rho \propto 1/r^{n-1}$

---

105. *Show that for the curve* $r(1 - \cos\theta) = 2a$, ρ^2 *varies as* r^3

>> $r(1 - \cos\theta) = 2a$

$\Rightarrow$ $\log r + \log(1 - \cos\theta) = \log 2a$

Differentiating w.r.t. θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1 - \cos\theta} = 0 \quad \text{or} \quad \frac{dr}{d\theta} = \frac{-r\sin\theta}{1 - \cos\theta}$$

ie., $$\frac{dr}{d\theta} = \frac{-2r\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -r\cot(\theta/2)$$

ie., $$r_1 = -r\cot(\theta/2)$$

Hence $r_2 = -r \cdot \dfrac{-1}{2}\operatorname{cosec}^2(\theta/2) - r_1\cot(\theta/2)$

ie., $$r_2 = \frac{r}{2}\operatorname{cosec}^2(\theta/2) + r\cot^2(\theta/2)$$

We have $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$$

$$\rho = \frac{\{r^2 + r^2 \cot^2(\theta/2)\}^{3/2}}{r^2 + 2r^2 \cot^2(\theta/2) - \frac{r^2}{2}\operatorname{cosec}^2(\theta/2) - r^2\cot^2(\theta/2)}$$

$$= \frac{(r^2)^{3/2}\{\operatorname{cosec}^2(\theta/2)\}^{3/2}}{r^2\left\{1 + \cot^2(\theta/2) - \frac{1}{2}\operatorname{cosec}^2(\theta/2)\right\}}$$

$$= \frac{r\operatorname{cosec}^3(\theta/2)}{1/2 \cdot \operatorname{cosec}^2(\theta/2)} = 2r\operatorname{cosec}(\theta/2)$$

$$\rho = 2r\operatorname{cosec}(\theta/2) \qquad \ldots (1)$$

But $r(1 - \cos\theta) = 2a$, by data

ie., $r \cdot 2\sin^2(\theta/2) = 2a$ or $\sin^2(\theta/2) = a/r$

$\therefore$ $\operatorname{cosec}(\theta/2) = \sqrt{r/a}$ and hence (1) becomes

$$\rho = 2r \cdot \sqrt{r/a} = 2\, r^{3/2}/\sqrt{a}$$

Thus $\rho^2 = 4r^3/a = (4/a) \cdot r^3 \Rightarrow \rho^2 \propto r^3$

Aliter : ***By the pedal form of*** ρ

The pedal equation of the given curve is $p^2 = ar$ [*Refer Example 73*]

Differentiating w.r.t. p,

$$2p = a\frac{dr}{dp} \text{ or } \frac{dr}{dp} = \frac{2p}{a} = \frac{2\sqrt{ar}}{a} = \frac{2\sqrt{r}}{\sqrt{a}}$$

Hence $\rho = r\dfrac{dr}{dp} = r \cdot \dfrac{2\sqrt{r}}{\sqrt{a}}$

ie., $\rho = \dfrac{2}{\sqrt{a}}(r)^{3/2}$

Thus $\rho^2 = (4/a) \cdot r^3 \Rightarrow \rho^2 \propto r^3$

106. *Find the radius of curvature of the curve $r = a\sin n\theta$ at the pole.*

>> $r = a\sin n\theta$

$$r_1 = an\cos n\theta,\ r_2 = -an^2\sin n\theta$$

At the pole we have $\theta = 0$. When $\theta = 0$: $r = 0$, $r_1 = an$, $r_2 = 0$

We have $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

$$\rho = \frac{(a^2n^2)^{3/2}}{2a^2n^2} = \frac{a^3n^3}{2a^2n^2} = \frac{an}{2}$$

Thus $\rho = an/2$ at the pole

---

107. *Show that at the point where the curve $r = a\theta$ intersects the curve $r = a/\theta$ their curvatures are in the ratio 3:1*

>> Equating the R.H.S of the two given equations

$r = a\theta$ and $r = a/\theta$ we have,

$$a\theta = \frac{a}{\theta} \text{ or } \theta^2 = 1 \quad \therefore \theta = \pm 1$$

Now $r = a\theta$, gives $r_1 = a$, $r_2 = 0$

At $\theta = +1$, $r = a$, $r_1 = a$, $r_2 = 0$ ... (1)

Also $r = a/\theta$ gives $r_1 = -a/\theta^2$, $r_2 = 2a/\theta^3$

At $\theta = +1$, $r = a$, $r_1 = -a$, $r_2 = 2a$ ... (2)

We have $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

From (1), $\rho = \dfrac{(a^2+a^2)^{3/2}}{a^2+2a^2} = \dfrac{(2a^2)^{3/2}}{3a^2} = \dfrac{2\sqrt{2}\,a}{3}$ ... (3)

From (2), $\rho = \dfrac{(a^2+a^2)^{3/2}}{a^2+2a^2-2a^2} = \dfrac{(2a^2)^{3/2}}{a^2} = \dfrac{2\sqrt{2}\,a}{1}$ ... (4)

Hence we have from (3) and (4) the ratio of the corresponding curvatures is given by $\dfrac{3/2\sqrt{2}\,a}{1/2\sqrt{2}\,a} = \dfrac{3}{1}$

Thus, **the curvatures are in the ratio 3:1**

---

108. *(a) Show that for the curve $r = a(1+\cos\theta)$, ρ^2/r is a constant*

(b) If ρ_1 and ρ_2 be the radii of curvatures at the extremities of the polar chord of the cardioide, show that $\rho_1^2+\rho_2^2 = 16a^2/9$

>> (a) $r = a(1+\cos\theta) \quad \Rightarrow \quad \log r = \log a + \log(1+\cos\theta)$

Differentiating w.r.t. θ, we have,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

$\therefore \quad r_1 = -r\tan(\theta/2)$

Hence $r_2 = -\dfrac{r}{2}\sec^2(\theta/2) - r_1\tan(\theta/2)$

ie., $\quad r_2 = -\dfrac{r}{2}\sec^2(\theta/2) + r\tan^2(\theta/2)$

We have $\quad \rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

$$\therefore \quad \rho = \frac{\{r^2+r^2\tan^2(\theta/2)\}^{3/2}}{r^2+2r^2\tan^2(\theta/2)+\dfrac{r^2}{2}\sec^2(\theta/2) - r^2\tan^2(\theta/2)}$$

$$= \frac{r^3\{\sec^2(\theta/2)\}^{3/2}}{r^2\left\{1+\tan^2(\theta/2)+\dfrac{1}{2}\sec^2(\theta/2)\right\}}$$

$$= \frac{r\sec^3(\theta/2)}{3/2\cdot\sec^2(\theta/2)} = \frac{2r}{3}\sec(\theta/2)$$

$$\rho = \frac{2r}{3}\sec(\theta/2) \qquad \ldots (1)$$

But $\quad r = a(1+\cos\theta) = a\cdot 2\cos^2(\theta/2)$

$\therefore \quad \sec^2(\theta/2) = \dfrac{2a}{r}$ or $\sec(\theta/2) = \dfrac{\sqrt{2a}}{\sqrt{r}}$

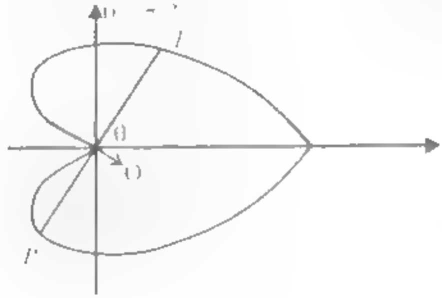
Hence (1) becomes $\rho = \dfrac{2r}{3}\dfrac{\sqrt{2a}}{\sqrt{r}}$ ie., $\rho = \dfrac{2}{3}\sqrt{2a\,r}$

$\therefore \quad \rho^2 = \dfrac{4}{9}(2\,a\,r)$ or $\dfrac{\rho^2}{r} = \dfrac{8a}{9}$ = constant.

Thus $\boldsymbol{\rho^2/r}$ **is a constant.**

(b) Let POP' be the polar chord (*chord passing through the pole*) of the cardioide $r = a(1+\cos\theta)$ Let ρ_1 and ρ_2 be the radii of curvatures at the point P and P' corresponding to the vectorial angles θ and $(\pi+\theta)$ respectively.

We have already obtained



$$\rho_1 = \frac{2r}{3}\sec(\theta/2)$$

[*first part of this example*]

$$\therefore \quad \rho_1^2 = \frac{4r^2}{9}\sec^2(\theta/2)$$

But $\quad r = a(1+\cos\theta) = 2a\cos^2(\theta/2)$

$$\therefore \quad r^2 = 4a^2\cos^4(\theta/2)$$

Hence $\rho_1^2 = \frac{4}{9}\cdot 4a^2\cos^4(\theta/2)\sec^2(\theta/2)$

ie., $$\rho_1^2 = \frac{16a^2}{9}\cos^2(\theta/2) \qquad \ldots(2)$$

Now changing θ to $(\pi+\theta)$ we have from (2)

$$\rho_2^2 = \frac{16a^2}{9}\cos^2\left(\frac{\pi+\theta}{2}\right) = \frac{16a^2}{9}\cos^2(\pi/2+\theta/2)$$

ie., $$\rho_2^2 = \frac{16a^2}{9}\sin^2(\theta/2) \qquad \ldots(3)$$

Thus we have by adding (2) and (3)

$$\rho_1^2 + \rho_2^2 = 16a^2/9 = \textbf{constant.}$$

---

109. *Show that the radius of curvature of the curve $pa^2 = r^3$ is $a^2/3r$*

Consider $pa^2 = r^3$ and differentiate w.r.t. p

$$a^2 = 3r^2\frac{dr}{dp} \quad \text{or} \quad \frac{dr}{dp} = \frac{a^2}{3r^2}$$

We have $\rho = r\dfrac{dr}{dp}$

$$\therefore \quad \rho = r\cdot\frac{a^2}{3r^2} = \frac{a^2}{3r}$$

Thus $\quad \rho = a^2/3r$

---

110. *Show that for the ellipse in the pedal form* $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2 b^2}$ *the radius of curvature at the point* (p, r) *is* $a^2 b^2/p$

>> Consider $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2 b^2}$

Differentiating w.r.t. p we get,

$$\frac{-2}{p^3} = \frac{-2r}{a^2 b^2} \frac{dr}{dp} \qquad \therefore \quad \frac{dr}{dp} = \frac{a^2 b^2}{p^3 r}$$

We have, $\rho = r \dfrac{dr}{dp}$

$$\rho = r \cdot \frac{a^2 b^2}{p^3 r} = \frac{a^2 b^2}{p^3}$$

Thus $\quad \rho = a^2 b^2/p^3$

Remark *Referring to Example 101, we have obtained the same result starting from the equation of the ellipse in the parametric form*

111. *Find the radius of curvature of the curve* $\theta = \dfrac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}(a/r)$, *at any point on it*

>> Differentiating the given equation w.r.t. r we have,

$$\frac{d\theta}{dr} = \frac{1}{a} \cdot \frac{2r}{2\sqrt{r^2 - a^2}} - \left\{ \frac{-1}{\sqrt{1 - (a/r)^2}} \cdot \frac{-a}{r^2} \right\}$$

$$= \frac{r}{a\sqrt{r^2 - a^2}} - \frac{r}{\sqrt{r^2 - a^2}} \cdot \frac{a}{r^2}$$

$$= \frac{1}{\sqrt{r^2 - a^2}} \left(\frac{r}{a} - \frac{a}{r} \right) = \frac{r^2 - a^2}{\sqrt{r^2 - a^2}\; a r}$$

ie., $$\frac{d\theta}{dr} = \frac{\sqrt{r^2 - a^2}}{ar} \qquad \ldots (1)$$

We prefer to find the pedal equation of the given curve and then apply the formula for ρ in the pedal form.

From (1) $\dfrac{1}{r} \dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{r^2 - a^2}} \qquad$ *ie.,* $\cot\phi = \dfrac{a}{\sqrt{r^2 - a^2}}$

Consider $p = r \sin\phi$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad ie., \frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{a^2}{r^2 - a^2}\right]$$

$$ie., \quad \frac{1}{p^2} = \frac{1}{r^2}\left[\frac{r^2}{r^2 - a^2}\right] \quad ie., \frac{1}{p} = \frac{1}{\sqrt{r^2 - a^2}}$$

$p = \sqrt{r^2 - a^2}$ is the pedal equation of the curve.

Differentiating w.r.t. p we get,

$$1 = \frac{2r}{2\sqrt{r^2 - a^2}}\frac{dr}{dp} \quad ie., \sqrt{r^2 - a^2} = r\frac{dr}{dp} = \rho$$

Thus $\rho = \sqrt{r^2 - a^2}$

---

112. *Prove that* $\rho = p + \dfrac{d^2 p}{d\psi^2}$ *with all the usual notations.*

>> We know that $p = r\sin\phi$

Now $\dfrac{dp}{d\psi} = \dfrac{dp}{dr}\dfrac{dr}{ds}\dfrac{ds}{d\psi}$ But $\dfrac{dr}{ds} = \cos\phi$ and $\dfrac{ds}{d\psi} = \rho = r\dfrac{dr}{dp}$

$$\therefore \quad \frac{dp}{d\psi} = \frac{dp}{dr}\cdot\cos\phi\cdot r\frac{dr}{dp} \quad \text{or} \quad \frac{dp}{d\psi} = r\cos\phi \qquad \ldots (2)$$

Squaring and adding (1) and (2) we get,

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2$$

Differentiating w.r.t p, we have,

$$2p + 2\frac{dp}{d\psi}\cdot\frac{d}{dp}\left(\frac{dp}{d\psi}\right) = 2r\frac{dr}{dp}$$

$$ie., \quad p + \frac{dp}{d\psi}\cdot\frac{d}{d\psi}\left(\frac{dp}{d\psi}\right)\cdot\frac{d\psi}{dp} = r\frac{dr}{dp}$$

But $\rho = r\dfrac{dr}{dp}$ and $p + \dfrac{d^2 p}{d\psi^2} = \rho$

Thus $\rho = p + \dfrac{d^2 p}{d\psi^2}$

Note *This form of expression for* ρ *is known as the tangential polar form*

Find the radius of curvature for the following curves [1 to 9]

1. $x^{2/3} + y^{2/3} = a^{2/3}$ at any point (x, y)
2. $xy^3 = a^4$ at the point (a, a)
3. $y^2 = x^3 + 8$ at the point $(-2, 0)$
4. $y = 4\sin x - \sin 2x$ at the point $(\pi/2, 4)$
5. $y = e^x$ at the point where the curve cuts the y-axis
6. $y^2 = a^2(a-x)/x$ where the curve cuts the x-axis.
7. $x = a \log \sec\theta,\ y = a(\tan\theta - \theta)$
8. $x = a(t - \sin t),\ y = a(1 - \cos t)$
9. $x = a\cos\theta,\ y = b\sin\theta$ at $(a/\sqrt{2},\ b/\sqrt{2})$
10. Show that for the curve $r^n = a^n \sin n\theta$, ρ varies inversely as r^{n-1}
11. Show that for the curve $r^2 \sec 2\theta = a^2$, $\rho = a^2/3r$
12. Show that for the curve $r\cos^2(\theta/2) = a$, ρ^2 varies as r^3
13. Obtain the pedal equation of the curve $r = a(1 - \cos\theta)$ and hence show that $\rho = (2/3)\sqrt{2ar}$
14. Using the pedal formula for ρ, prove that $\rho = r^3/a^2$ for the curve $r^2 = a^2 \sec 2\theta$
15. Show that for the curve $p^2 = ar$, ρ^2 varies as r^3

ANSWERS

1. $3(axy)^{1/3}$	2. $5\sqrt{10}\,a/6$	3. 6
4. $5\sqrt{5}/4$	5. $2\sqrt{2}$	6. $a/2$
7. $a\tan\theta\sec\theta$	8. $4a\sin(t/2)$	9. $(a^2+b^2)^{3/2}/ab$

PART

Unit - III

So far we have discussed various concepts related to a function of one independent variable only. That is, discussion related to $y = f(x)$.

Practically we do come across with quantities whose value depend on more than one independent variable.

For example the area (A) of a rectangle depends on its length (l) and breadth (b). That is $A = f(l, b)$ Volume (V) of a parallelopiped depends on its length (l), breadth (b) and height (h). That is $V = f(l, b, h)$.

We are conversant with the differentiation of a function of one independent variable and this topic is again the discussion of known concepts of differentiation in a broader perspective.

Precisely, this topic deals with the differentiation of a function of many independent variables

Let u be a function of two independent variables x and y

ie., $u = f(x, y)$.

Recapitulating the definition of $\frac{dy}{dx}$ in the case of $y = f(x)$,

[*ordinary derivative to be specific*] the partial derivatives of u *w.r.t* x and also *w.r.t* y are defined in a similar fashion.

Here goes the definitions.

$$\frac{\partial u}{\partial x} = \underset{\delta x \to 0}{lt} \frac{f(x+\delta x, y) - f(x, y)}{\delta x} \quad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \underset{\delta y \to 0}{lt} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \quad \ldots (2)$$

[*"∂" is the symbol of partial derivative usually read as "del" and it should not be written as "δ" (delta) an acquainted symbol*]

Thus (1) is *called the partial derivative of* u *w.r.t* x and (2) is called the partial derivative of u *w.r.t* y.

It should be observed that in the equation (1) x is varying whereas y is remaining constant. Also in the equation (2) y is varying whereas x is remaining constant.

The derivative of u *w.r.t.* x treating y as a constant is called as the partial derivative of u *w.r.t.* x and is denoted by $\frac{\partial u}{\partial x}$ or u_x

Similarly the derivative of u *w.r.t.* y treating x as a constant is called the partial derivative of u *w.r.t.* y and is denoted by $\frac{\partial u}{\partial y}$ or u_y

The product rule, quotient rule and the function of a function rule continues to hold good here also.

Observe the following comparisons.

Ordinary derivative

1. $y = 3x^2 + 6x + 7$

$$\frac{dy}{dx} = 6x + 6 + 0 = 6x + 6$$

2. $y = e^{4x+3}$

$$\frac{dy}{dx} = e^{4x+3} \cdot \frac{d}{dx}(4x+3) = e^{4x+3} \cdot 4 = 4e^{4x+3}$$

3. $y = \sin 5x$

$$\frac{dy}{dx} = \cos 5x \cdot 5 = 5\cos 5x$$

4. $y = \tan^{-1}(2/x)$

$$\frac{dy}{dx} = \frac{1}{1+(2/x)^2} \cdot \frac{d}{dx}\left(\frac{2}{x}\right) = \frac{x^2}{x^2+4} \cdot \left(-\frac{2}{x^2}\right) = -\frac{2}{x^2+4}$$

Partial derivatives

1. $u = 3x^2 y + 6xy^2 + 7$

$$\frac{\partial u}{\partial x} = (6x)\, y + 6 \cdot 1 \cdot y^2 + 0 = 6xy + 6y^2$$

[y is treated as constant]

$$\frac{\partial u}{\partial y} = 3x^2 \cdot 1 + 6x \cdot 2y + 0 = 3x^2 + 12xy$$

[x is treated as constant]

2. $u = e^{4x+3y}$

$$\frac{\partial u}{\partial x} = e^{4x+3y} \cdot \frac{\partial}{\partial x}(4x+3y) = e^{4x+3y} \cdot (4+0) = 4e^{4x+3y}$$

$$\frac{\partial u}{\partial y} = e^{4x+3y} \cdot \frac{\partial}{\partial y}(4x+3y) = e^{4x+3y} \cdot (0+3) = 3e^{4x+3y}$$

3. $u = \sin(xy)$

$$\frac{\partial u}{\partial x} = \cos(xy)\frac{\partial}{\partial x}(xy) = \cos(xy) \cdot y$$

$$\frac{\partial u}{\partial y} = \cos(xy)\frac{\partial}{\partial y}(xy) = \cos(xy) \cdot x$$

4. $u = \tan^{-1}(y/x)$, $\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$,

$$i.e., \; = \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

Ordinary derivative	Partial derivatives
5. $\frac{d}{dx}[f(y)] = f'(y)\frac{dy}{dx}$	5. If r is a function of x and y, $\frac{\partial}{\partial x}[f(r)] = f'(r)\frac{\partial r}{\partial x}$, $\frac{\partial}{\partial y}[f(r)] = f'(r)\frac{\partial r}{\partial y}$

General principle of partial differentiation

Given a function of many independent variables, the derivative of this function with *respect to a particular* independent variable, keeping (*treating*) all other independent variables as constants is the general principle of partial differentiation.

Higher order partial derivatives

These are also analogous with the higher order ordinary derivatives.

Let us suppose that $u = f(x, y)$. The development of higher order partial derivatives is as exhibited below.

$$u = f(x, y)$$

First order partial derivatives

$$u_x = \frac{\partial u}{\partial x} \qquad\qquad u_y = \frac{\partial u}{\partial y}$$

Second order partial derivatives

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \qquad\qquad u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

$$u_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} \leftarrow \text{Mixed partial derivatives} \rightarrow u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y}$$

and so-on

It is very important to note that;

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} \quad \text{or} \quad u_{yx} = u_{xy}$$

WORKED PROBLEMS

Type - 1 Direct partial derivatives

Given an explicit function of more than one independent variable we find the required partial derivatives just keeping in mind the *general principle of partial differentiation* stated earlier along with the well acquainted rules of differentiation. The following note on symmetric functions will be highly useful for certain problems.

Note **Symmetric function** : *A function $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x)$ and a function $f(x, y, z)$ is said to be symmetric if $f(x, y, z) = f(y, z, x) = f(z, x, y)$. In general we can say that a function of several variables is symmetric if the function remains unchanged (invariant) when the variables are cyclically rotated. Observe the following examples*

(i) $\quad x + y,\ x^2 + y^2,\ \dfrac{x^2 + y^2}{x + y},\ x^2 + xy + y^2,\ \log\sqrt{x^2 + y^2}$ *etc.*

are symmetric functions of two variables as it can be easily seen that when x is replaced by y and y by x the functions remain the same.

(ii) $\quad x^2 + y^2 + z^2,\ xy + yz + zx,\ x/y + y/z + z/x,\ \log(x + y + z),$

$\quad x^3 + y^3 + z^3 - 3xyz$ *etc.*

are symmetric functions of three variables

It is very important to note that, *if we have a symmetric function of three variables say $u = f(x, y, z)$ then by just computing u_x or u_{xx} or u_{xy} we can simply writedown easily the other partial derivatives (u_y, u_z) or (u_{yy}, u_{zz}) or (u_{yz}, u_{zx}) by simple guess work. There is no need to show the working of similar computation of partial derivatives*

We have $u = x^3 - 3xy^2 + x + e^x \cos y + 1$

$\therefore \quad \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 + e^x \cos y \qquad$ *(y is treated as constant.)*

Differentiating this *w.r.t* x partially again,

$\dfrac{\partial^2 u}{\partial x^2} = 6x + e^x \cos y \qquad$ *(Again y is treated as constant.)*

Next, $\dfrac{\partial u}{\partial y} = -6xy - e^x \sin y \qquad$ *(x is treated as constant.)*

Differentiating this *w.r.t* y partially again,

$\dfrac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y \qquad$ *(Again x is treated as constant.)*

Now $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y = 0$

This proves the desired result.

>> We have $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$

$\therefore \quad \frac{\partial u}{\partial x} = e^{-2\pi^2 t} (\pi \cos \pi x) \sin \pi y \quad \ldots$ (*t* & *y* *are treated as constants*)

Differentiating this *w.r.t* x partially again,

$$\frac{\partial^2 u}{\partial x^2} = e^{-2\pi^2 t} (-\pi^2 \sin \pi x) \sin \pi y = -\pi^2 u$$

$\frac{\partial u}{\partial y} = e^{-2\pi^2 t} \sin \pi x (\pi \cos \pi y) \quad \ldots$ (*t* & *x* *are treated as constants*)

Differentiating this *w.r.t* y partially again,

$$\frac{\partial^2 u}{\partial y^2} = e^{-2\pi^2 t} \sin \pi x (-\pi^2 \sin \pi y) = -\pi^2 u$$

Thus $\quad$ L.H.S $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\pi^2 u - \pi^2 u = -2\pi^2 u \quad$ (1)

Also $\quad \frac{\partial u}{\partial t} = e^{-2\pi^2 t} (-2\pi^2) \sin \pi x \sin \pi y \quad \ldots$ (*x* & *y* *are treated as constants*)

Thus $\quad$ R.H.S $= \frac{\partial u}{\partial t} = -2\pi^2 u \quad \ldots$ (2)

$\therefore$ from (1) and (2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

>> $\quad u = \log(x^2 + y^2) - \log(x + y)$

$\therefore \quad u_x = \frac{1}{x^2 + y^2} \cdot 2x - \frac{1}{x + y} \cdot 1$

and $\quad u_y = \frac{1}{x^2 + y^2} \cdot 2y - \frac{1}{x + y} \cdot 1$

Now, $\quad x\,u_x + y\,u_y = \dfrac{2x^2}{x^2+y^2} - \dfrac{x}{x+y} + \dfrac{2y^2}{x^2+y^2} - \dfrac{y}{x+y}$

$$= \frac{2(x^2+y^2)}{x^2+y^2} - \frac{(x+y)}{x+y} = 2-1 = 1$$

Thus $\quad x\,u_x + y\,u_y = 1$

---

4. If $u = e^{ax-by}\sin(ax+by)$ show that $b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y} = 2abu$

>> $\quad u = e^{ax-by}\sin(ax+by)$

$$\frac{\partial u}{\partial x} = e^{ax-by}\cos(ax+by)\cdot a + a\cdot e^{ax-by}\sin(ax+by)$$

ie, $\quad \dfrac{\partial u}{\partial x} = a\,e^{ax-by}\cos(ax+by) + a\,u \qquad \ldots(1)$

Also $\quad \dfrac{\partial u}{\partial y} = e^{ax-by}\cos(ax+by)\cdot b + (-b)\,e^{ax-by}\sin(ax+by)$

ie, $\quad \dfrac{\partial u}{\partial y} = b\,e^{ax-by}\cos(ax+by) - b\,u \qquad \ldots(2)$

Now $\quad b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y}$ by using (1) and (2) becomes

$$= abe^{ax-by}\cos(ax+by) + abu - abe^{ax-by}\cos(ax+by) + abu$$

$$= 2abu$$

Thus $\quad b\dfrac{\partial u}{\partial x} - a\dfrac{\partial u}{\partial y} = 2abu$

---

5. If $u = e^{-c^2p^2t}(a\cos px + b\sin px)$, show that $u_t = c^2 u_{xx}$

>> $\quad u_t = e^{-c^2p^2t}(-c^2p^2)(a\cos px + b\sin px)$

ie, $\quad u_t = -c^2p^2u \qquad \ldots(1)$

Also $\quad u_x = e^{-c^2p^2t}(-ap\sin px + bp\cos px)$

$$u_{xx} = e^{-c^2p^2t}(-ap^2\cos px - bp^2\sin px)$$

ie, $\quad u_{xx} = -p^2 e^{-c^2p^2t}(a\cos px + b\sin px)$

or $\quad u_{xx} = -p^2 u$ by using the data.

$\therefore \quad c^2 u_{xx} = -c^2 p^2 u \qquad \ldots (2)$

Thus from (1) and (2) we have $u_t = c^2 u_{xx}$

---

6. *If* $u = \dfrac{1}{r^2}\cos 2\theta$, *prove that* $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$

>> $$\frac{\partial u}{\partial r} = -\frac{2}{r^3}\cos 2\theta \qquad \ldots (1)$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{6}{r^4}\cos 2\theta \qquad \ldots (2)$$

Also $\dfrac{\partial u}{\partial \theta} = -\dfrac{2}{r^2}\sin 2\theta$ and $\dfrac{\partial^2 u}{\partial \theta^2} = -\dfrac{4}{r^2}\cos 2\theta$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{4}{r^4}\cos 2\theta \qquad \ldots (3)$$

Thus from (1), (2) and (3) $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

$$= \frac{6}{r^4}\cos 2\theta - \frac{2}{r^4}\cos 2\theta - \frac{4}{r^4}\cos 2\theta = 0$$

This proves the desired result.

---

7. *If* $z = \sinh^{-1}(x/y)$, *show that* $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$

>> $$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1+(x^2/y^2)}}\cdot\frac{1}{y} = \frac{y}{\sqrt{y^2+x^2}}\cdot\frac{1}{y} = \frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1+(x^2/y^2)}}\cdot\frac{-x}{y^2} = \frac{y}{\sqrt{y^2+x^2}}\cdot\frac{-x}{y^2} = \frac{-x}{y\sqrt{x^2+y^2}}$$

Now $\quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{x}{\sqrt{x^2+y^2}} - \dfrac{x}{\sqrt{x^2+y^2}} = 0$

This proves the desired result.

---

8. *If* $u = \tan^{-1}(y/x)$ *verify that*

(a) $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial^2 u}{\partial x\,\partial y}$ (b) $\dfrac{\partial^3 u}{\partial y^2\,\partial x} = \dfrac{\partial^3 u}{\partial x\,\partial y^2} = \dfrac{\partial^3 u}{\partial y\,\partial x\,\partial y}$

>> We have $u = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

(a) Now $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$$= \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(-1)-(-y)\,2y}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} \quad \text{. by quotient rule}$$

Thus $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ... (1)

Also $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$\therefore$ $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ... (2)

From (1) & (2) $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$

(b) $\dfrac{\partial^3 u}{\partial y^2\,\partial x} = \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left\{\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)\right\}$

$$= \frac{\partial}{\partial y}\left\{\frac{y^2-x^2}{(x^2+y^2)^2}\right\} \text{ ... by using (1)}$$

$$\frac{(x^2+y^2)^2 \cdot (2y) - (y^2-x^2)\,2\,(x^2+y^2)\,2y}{(x^2+y^2)^4} \quad \text{.. by quotient rule}$$

$$= \frac{2y\,(x^2+y^2)\left[\,(x^2+y^2) - 2\,(y^2-x^2)\,\right]}{(x^2+y^2)^4}$$

Thus $\dfrac{\partial^3 u}{\partial y^2\,\partial x} = \dfrac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$ ... (3)

Also
$$\frac{\partial^3 u}{\partial x\,\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial x}\left\{\frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2}\right\} = \frac{\partial}{\partial x}\left\{-\frac{2xy}{(x^2+y^2)^2}\right\}$$

$$= -\frac{(x^2+y^2)^2(-2y) + 2xy\cdot 2\,(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= -\frac{2y\,(x^2+y^2)\left[-(x^2+y^2)+4x^2\right]}{(x^2+y^2)^4}$$

$\therefore$ $\dfrac{\partial^3 u}{\partial x\,\partial y^2} = \dfrac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$ ... (4)

Also
$$\frac{\partial^3 u}{\partial y\,\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left\{\frac{y^2-x^2}{(x^2+y^2)^2}\right\} \quad \text{... by using (2)}$$

$$= \frac{(x^2+y^2)^2\cdot(2y) - (y^2-x^2)\,2\,(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{2y\,(x^2+y^2)\,[x^2+y^2-2y^2+2x^2]}{(x^2+y^2)^4}$$

Thus $\dfrac{\partial^3 u}{\partial y\,\partial x\,\partial y} = \dfrac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$ ... (5)

From (3), (4) and (5) the result follows

9. *If* $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$ *show that* $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$

>> We have $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

By data, $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$

$$\frac{\partial z}{\partial y} = x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} - \left\{ y^2 \cdot \frac{1}{1+(x/y)^2} \cdot -\frac{x}{y^2} + 2y \tan^{-1}(x/y) \right\}$$

(We have used the product rule while differentiating the second term)

ie., $$\frac{\partial z}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{x y^2}{x^2+y^2} - 2y \tan^{-1}(x/y)$$

ie., $$\frac{\partial z}{\partial y} = \frac{x(x^2+y^2)}{(x^2+y^2)} - 2y \tan^{-1}(x/y)$$

$\therefore$ $$\frac{\partial z}{\partial y} = x - 2y \tan^{-1}(x/y)$$

Now differentiating *w.r.t* x partially we have,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 1 - 2y\left[\frac{1}{1+(x/y)^2} \cdot \frac{1}{y}\right]$$

or $$\frac{\partial^2 z}{\partial x\, \partial y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Thus $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{x^2-y^2}{x^2+y^2}$

---

10. *If* $u = \tan^{-1}\left[\dfrac{xy}{\sqrt{1+x^2+y^2}}\right]$, *show that* $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{1}{(1+x^2+y^2)^{3/2}}$

>> $$u = \tan^{-1}\left[\frac{xy}{\sqrt{1+x^2+y^2}}\right] \text{ by data.}$$

$$\frac{\partial u}{\partial y} = \left(\frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}}\right) \cdot \frac{\partial}{\partial y}\left[\frac{xy}{\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2} \cdot x \left\{ \frac{\sqrt{1+x^2+y^2} \cdot 1 - \dfrac{y}{2\sqrt{1+x^2+y^2}} \times 2y}{(1+x^2+y^2)} \right\}$$

$$\frac{\partial u}{\partial y} = \frac{x}{(1+x^2)(1+y^2)} \cdot \left\{ \frac{1+x^2+y^2-y^2}{\sqrt{1+x^2+y^2}} \right\}$$

ie., $$\frac{\partial u}{\partial y} = \frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}$$

Now, $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{1}{1+y^2} \cdot \frac{\partial}{\partial x}\left\{\frac{x}{\sqrt{1+x^2+y^2}}\right\}$$

ie., $$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{1+y^2} \cdot \left\{ \frac{\sqrt{1+x^2+y^2} \cdot 1 - x \cdot \frac{2x}{2\sqrt{1+x^2+y^2}}}{1+x^2+y^2} \right\}$$

$$= \frac{1}{1+y^2} \cdot \left\{ \frac{1+x^2+y^2-x^2}{(1+x^2+y^2)^{3/2}} \right\}$$

$$= \frac{1}{1+y^2} \cdot \frac{1+y^2}{(1+x^2+y^2)^{3/2}}$$

Thus $$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$$

Verify that $u_{xy} = u_{yx}$ for the following functions.

11. $u = \sin^{-1}(y/x)$ **12.** $u = x^y$ **13.** $u = e^x(x\cos y - y\sin y)$

11. $u = \sin^{-1}(y/x)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{-y}{x^2} = \frac{x}{\sqrt{x^2-y^2}} \cdot \frac{-y}{x^2}, \quad i.e., \quad \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(y/x)^2}} \cdot \frac{1}{x} = \frac{x}{\sqrt{x^2-y^2}} \cdot \frac{1}{x} = \frac{1}{\sqrt{x^2-y^2}}$$

Now $$\frac{\partial^2 u}{\partial x\,\partial y} = u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{x^2-y^2}}\right\}$$

ie., $$= \frac{\partial}{\partial x}(x^2-y^2)^{-1/2} = -\frac{1}{2}(x^2-y^2)^{-3/2} \cdot 2x$$

Thus $$u_{xy} = \frac{-x}{(x^2-y^2)^{3/2}} \qquad \ldots (1)$$

Also $\quad \dfrac{\partial^2 u}{\partial y\, \partial x} = u_{yx} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(-\dfrac{y}{x\sqrt{x^2-y^2}}\right)$

ie., $\quad = -\dfrac{1}{x}\left\{\dfrac{\sqrt{x^2-y^2}\cdot 1 - y\cdot\dfrac{-2y}{2\sqrt{x^2-y^2}}}{x^2-y^2}\right\}$

$$= -\frac{1}{x}\left[\frac{x^2-y^2+y^2}{(x^2-y^2)^{3/2}}\right] = -\frac{x}{(x^2-y^2)^{3/2}}$$

Thus $\quad u_{yx} = \dfrac{-x}{(x^2-y^2)^{3/2}}$

From (1) and (2) $u_{xy} = u_{yx}$. **The result is verified**

12. $\quad u = x^y$

$\dfrac{\partial u}{\partial x} = y\,x^{y-1}$ $\quad$ *Treating y as a constant we have used* $\dfrac{d}{dx}(x^n) = n\,x^{n-1}$

$\dfrac{\partial u}{\partial y} = x^y \log x$ $\quad$ *Treating x as a constant we have used* $\dfrac{d}{dx}(a^x) = a^x \log a$

$u_{xy} = \dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}(x^y \log x)$

ie., $\quad = x^y \cdot \dfrac{1}{x} + y\,x^{y-1}\cdot \log x$ $\quad \ldots$ *(by product rule)*

$\quad = x^{y-1} + y\,x^{y-1}\log x$

Thus $\quad u_{xy} = x^{y-1}(1 + y\log x)$ $\quad \ldots (1)$

Also $\quad u_{yx} = \dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}(y\,x^{y-1})$

ie., $\quad = y\cdot x^{y-1}\log x + x^{y-1}\cdot 1 = x^{y-1}(y\log x + 1)$

Thus $\quad u_{yx} = x^{y-1}(1 + y\log x)$ $\quad \ldots (2)$

From (1) and (2) $u_{xy} = u_{yx}$. **The result is verified.**

13. $u = e^x(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^x \cdot \cos y + e^x(x\cos y - y\sin y) \quad \textit{...by product rule.}$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = -e^x(x\sin y + y\cos y + \sin y)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left\{e^x(x\sin y + y\cos y + \sin y)\right\}$$

ie., $\quad = -\left\{e^x \cdot \sin y + e^x(x\sin y + y\cos y + \sin y)\right\}$

Thus $\quad u_{xy} = -e^x(2\sin y + x\sin y + y\cos y) \qquad \ldots(1)$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left\{e^x(\cos y + x\cos y - y\sin y)\right\}$$

ie., $\quad = e^x(-\sin y - x\sin y - y\cos y - \sin y)$

Thus $\quad u_{yx} = -e^x(2\sin y + x\sin y + y\cos y) \qquad \ldots(2)$

From (1) and (2), $u_{xy} = u_{yx}$. **The result is verified.**

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14. *If* $z = \tan(y+ax) + (y-ax)^{3/2}$ *show that* $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$

>> We have $z = \tan(y+ax) + (y-ax)^{3/2}$

$$\frac{\partial z}{\partial x} = \sec^2(y+ax) \cdot a + \frac{3}{2}(y-ax)^{1/2} \cdot (-a)$$

Differentiating *w.r.t* x again partially,

$$\frac{\partial^2 z}{\partial x^2} = 2\sec(y+ax) \cdot \sec(y+ax)\tan(y+ax) \cdot a^2 + \frac{3}{2} \cdot \frac{1}{2}(y-ax)^{-1/2} \cdot a^2$$

$$\frac{\partial^2 z}{\partial x^2} = a^2\left\{2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}\right\} \qquad \ldots(1)$$

Now, $\quad \frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$

Also $\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 2\sec(y+ax) \cdot \sec(y+ax) \cdot \tan(y+ax) + \frac{3}{2} \cdot \frac{1}{2}(y-ax)^{-1/2}$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2} \qquad \ldots (2)$$

Using (2) in (1) we have $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$

Thus $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2} = 0$

---

15. *If* $v = e^{a\theta}\cos(a\log r)$, *prove that* $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

\>> $v = e^{a\theta}\cos(a\log r)$

$$\frac{\partial v}{\partial r} = -e^{a\theta}\sin(a\log r)\cdot\frac{a}{r} \qquad \ldots (1)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}\right) = -e^{a\theta}\left[\sin(a\log r)\cdot -\frac{a}{r^2}\right] + \cos(a\log r)\cdot\frac{a^2}{r^2}$$

ie., $$\frac{\partial^2 v}{\partial r^2} = \frac{a}{r^2}e^{a\theta}\sin(a\log r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\log r) \qquad \ldots (2)$$

Next, $\dfrac{\partial v}{\partial \theta} = ae^{a\theta}\cos(a\log r)$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta}\right) = \frac{\partial^2 v}{\partial \theta^2} = a^2 e^{a\theta}\cos(a\log r) \qquad \ldots (3)$$

Consider $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2}$ by using (1), (2) and (3)

$$= \left\{\frac{a}{r^2}e^{a\theta}\sin(a\log r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\log r)\right\}$$

$$+ \left\{-\frac{a}{r^2}e^{a\theta}\sin(a\log r)\right\} + \left\{\frac{a^2}{r^2}e^{a\theta}\cos(a\log r)\right\} = 0$$

Thus $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

---

16. *If* $\theta = t^n e^{-r^2/4t}$ *find the value of* n *such that* $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left[r^2\dfrac{\partial\theta}{\partial r}\right] = \dfrac{\partial\theta}{\partial t}$

>> We have $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial\theta}{\partial t} = t^n e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right) + n\,t^{n-1} e^{-r^2/4t} \quad \text{...by product rule.}$$

ie., $$\frac{\partial\theta}{\partial t} = \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} + n\,t^{n-1} e^{-r^2/4t} \qquad \ldots(1)$$

Next $$\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t}\cdot\left(-\frac{2r}{4t}\right) = -\frac{1}{2} t^{n-1}\, r\, e^{-r^2/4t}$$

$$\therefore \quad r^2\frac{\partial\theta}{\partial r} = -\frac{1}{2} t^{n-1} r^3 e^{-r^2/4t}$$

Now $$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{t^{n-1}}{2}\left\{r^3 e^{-r^2/4t}\cdot\left(-\frac{2r}{4t}\right) + 3r^2 e^{-r^2/4t}\right\}$$

$$= \frac{1}{4} r^4 t^{n-2} e^{-r^2/4t} - \frac{3}{2} t^{n-1} r^2 e^{-r^2/4t}$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} - \frac{3}{2} t^{n-1} e^{-r^2/4t} \qquad \ldots(2)$$

Using (1) & (2) in the equation, $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right) = \dfrac{\partial\theta}{\partial t}$ we have,

$$\frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} - \frac{3}{2} t^{n-1} e^{-r^2/4t} = \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} + n\,t^{n-1} e^{-r^2/4t}$$

or $$-\frac{3}{2} t^{n-1} e^{-r^2/4t} = n\,t^{n-1} e^{-r^2/4t} \Rightarrow -\frac{3}{2} = n$$

Thus the required value of $n = -3/2$.

---

17. *If* $u = e^{ax+by} f(ax-by)$, *prove that* $b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y} = 2abu$

>> $u = e^{ax+by} f(ax-by)$, by data.

$$\frac{\partial u}{\partial x} = e^{ax+by}\cdot f'(ax-by)\,a + a\,e^{ax+by} f(ax-by)$$

or $$\frac{\partial u}{\partial x} = a\,e^{ax+by} f'(ax-by) + a\,u \qquad \ldots(1)$$

Next, $$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)\cdot(-b) + b\,e^{ax+by} f(ax-by)$$

or $\quad \dfrac{\partial u}{\partial y} = -b\, e^{ax+by} f'(ax-by) + bu \qquad \ldots (2)$

Now consider L.H.S $= b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y}$ by using (1) and (2)

$$= b\left\{a e^{ax+by} f'(ax-by) + au\right\} + a\left\{-b e^{ax+by} f'(ax-by) + bu\right\}$$

$$= ab e^{ax+by} f'(ax-by) + abu - ab e^{ax+by} f'(ax-by) + abu$$

$$2abu = \text{R.H.S}$$

Thus $\quad \boldsymbol{b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y} = 2\,abu.}$

---

18. *If* $u = f(x+ct) + g(x-ct)$ *prove that* $u_{tt} = c^2 u_{xx}$

>> We have $u = f(x+ct) + g(x-ct)$

$$u_x = \frac{\partial u}{\partial x} = f'(x+ct)\cdot 1 + g'(x-ct)\cdot 1$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = f''(x+ct) + g''(x-ct) \qquad \ldots (1)$$

$$u_t = \frac{\partial u}{\partial t} = f'(x+ct)\cdot(c) + g'(x-ct)\cdot(-c)$$

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = f''(x+ct)\cdot(c^2) + g''(x-ct)(c^2)$$

ie., $\quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2} = c^2\left[f''(x+ct) + g''(x-ct)\right] \qquad \ldots (2)$

Using (1) in (2) we have $u_{tt} = c^2 u_{xx}$

---

19. *If* $u = \dfrac{1}{r}\left[f(r-at) + \phi(r+at)\right]$ *show that* $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{a^2}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$

>> We have $u = \dfrac{1}{r}\left[f(r-at) + \phi(r+at)\right]$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{1}{r}\left[f'(r-at)\cdot(-a) + \phi'(r+at)\cdot a\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{1}{r}\left[f''(r-at)\,a^2 + \phi''(r+at)\,a^2\right]$$

ie., $$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r}\left[f''(r-at)+\phi''(r+at)\right] \qquad \ldots(1)$$

Next $$\frac{\partial u}{\partial r} = \frac{1}{r}\left[f'(r-at)\cdot 1+\phi'(r+at)\cdot 1\right]+\left[f(r-at)+\phi(r+at)\right]\cdot\left(-\frac{1}{r^2}\right)$$

$\therefore$ $$r^2\frac{\partial u}{\partial r} = r\left[f'(r-at)+\phi'(r+at)\right]-\left[f(r-at)+\phi(r+at)\right]$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = r\left[f''(r-at)+\phi''(r+at)\right]+\left[f'(r-at)+\phi'(r+at)\right]$$
$$-\left[f'(r-at)+\phi'(r+at)\right]$$
$$= r\left[f''(r-at)+\phi''(r+at)\right]$$

$\therefore$ $$\frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{a^2}{r}\left[f''(r-at)+\phi''(r+at)\right] \qquad \ldots(2)$$

From (1) & (2) $$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

20. If $z = f_1(y-3x)+f_2(y+2x)+\sin x - y\cos x$

show that $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\,\partial y}-6\dfrac{\partial^2 z}{\partial y^2} = y\cos x$

>> $$z = f_1(y-3x)+f_2(y+2x)+\sin x - y\cos x$$

$$\frac{\partial z}{\partial x} = f_1'(y-3x)\cdot(-3)+f_2'(y+2x)\cdot 2+\cos x+y\sin x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_1''(y-3x)\cdot 9+f_2''(y+2x)\cdot 4-\sin x+y\cos x$$

ie., $$\frac{\partial^2 z}{\partial x^2} = 9f_1''(y-3x)+4f_2''(y+2x)-\sin x+y\cos x \qquad \ldots(1)$$

$$\frac{\partial z}{\partial y} = f_1'(y-3x)+f_2'(y+2x)-\cos x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_1''(y-3x)+f_2''(y+2x) \qquad \ldots(2)$$

Also $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_1''(y-3x)\cdot(-3)+f_2''(y+2x)\cdot 2+\sin x$$

ie., $$\frac{\partial^2 z}{\partial x\,\partial y} = -3f_1''(y-3x)+2f_2''(y+2x)+\sin x \qquad \ldots (3)$$

Consider $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\,\partial y}-6\dfrac{\partial^2 z}{\partial y^2}$ using (1), (2) and (3).

$$= \left[\,9f_1''(y-3x)+4f_2''(y+2x)-\sin x+y\cos x\,\right]$$
$$+\left[\,-3f_1''(y-3x)+2f_2''(y+2x)+\sin x\,\right]-6\left[\,f_1''(y-3x)+f_2''(y+2x)\,\right]$$
$$= 9f_1''(y-3x)-9f_1''(y-3x)+6f_2''(y+2x)-6f_2''(y+2x)+y\cos x$$
$$-\,y\cos x$$

Thus $$\boldsymbol{\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\,\partial y}-6\frac{\partial^2 z}{\partial y^2} = y\cos x}$$

Problems on symmetric functions

21. *If* $u = \log\sqrt{x^2+y^2+z^2}$, *show that* $(x^2+y^2+z^2)\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right.$

>> By data $u = \log\sqrt{x^2+y^2+z^2} = \dfrac{1}{2}\log(x^2+y^2+z^2)$

The given u is a symmetric function of $x, y, z,$

(It is enough if we compute only one of the required partial derivative)

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$$

ie., $$= \frac{(x^2+y^2+z^2)\,1-x\cdot 2x}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} \qquad \ldots (1)$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2} \qquad \ldots (2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \qquad \ldots (3)$$

Adding (1), (2), and (3) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

Thus $\quad (x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

22. *If* $u = \log(\tan x + \tan y + \tan z)$, *show that* $\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = 2$

\>> $\quad u = \log(\tan x + \tan y + \tan z)$ is a symmetric function.

$$u_x = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\therefore \quad \sin 2x\, u_x = \frac{(2\sin x\cos x)\sec^2 x}{\tan x + \tan y + \tan z}$$

or $\quad \sin 2x\, u_x = \dfrac{2\tan x}{\tan x + \tan y + \tan z} \qquad \ldots (1)$

Similarly $\quad \sin 2y\, u_y = \dfrac{2\tan y}{\tan x + \tan y + \tan z} \qquad \ldots (2)$

$$\sin 2z\, u_z = \frac{2\tan z}{\tan x + \tan y + \tan z} \qquad \ldots (3)$$

Adding (1),(2) and (3) we get,

$$\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

Thus $\quad \mathbf{\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = 2}$

23. *If* $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ *then show that* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

\>> $\quad u = (x^2+y^2+z^2)^{-1/2}$ is a symmetric function of x, y, z,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x = -(x^2+y^2+z^2)^{-3/2}\cdot x$$

$$\therefore \quad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\left\{(x^2+y^2+z^2)^{-3/2}\cdot x\right\}$$

$$= -\left\{(x^2+y^2+z^2)^{-3/2}\cdot 1 + x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\cdot 2x\right\}$$

$$= -\left\{(x^2+y^2+z^2)^{-3/2} - 3x^2(x^2+y^2+z^2)^{-5/2}\right\}$$

ie, $$\frac{\partial^2 u}{\partial x^2} = 3x^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \qquad (1)$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = 3y^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \qquad \ldots (2)$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \qquad \ldots (3)$$

Adding the results (1), (2) and (3) we have,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$$

Thus $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

24. *If $u = \log(x^3+y^3+z^3-3xyz)$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$*

and hence show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

\>> $u = \log(x^3+y^3+z^3-3xyz)$ is a symmetric function.

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \qquad (1)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz} \qquad (2)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \qquad (3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x^3+y^3+z^3-3xyz)}$$

Recalling a standard elementary result,

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

we have,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\,(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)\,(x^2+y^2+z^2-xy-yz-zx)}$$

Thus $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x+y+z}$

Further $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ by using the earlier result.}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

Thus $\quad \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$

25. *If* $z(x+y) = x^2+y^2$, *show that* $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$

\>> $\quad z = \dfrac{x^2+y^2}{x+y}$ by data, is a symmetric function of x, y

$$\frac{\partial z}{\partial x} = \frac{(x+y)\,2x-(x^2+y^2)\,1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2} \qquad \ldots (1)$$

Similarly $\quad \dfrac{\partial z}{\partial y} = \dfrac{y^2+2xy-x^2}{(x+y)^2} \qquad (2)$

Now $\left[\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right]=\dfrac{(x^2+2xy-y^2)-(y^2+2xy-x^2)}{(x+y)^2}$

$$=\frac{2(x^2-y^2)}{(x+y)^2}=\frac{2(x-y)(x+y)}{(x+y)^2}=\frac{2(x-y)}{(x+y)}$$

$$\therefore \qquad \left[\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]^2=4\,\frac{(x-y)^2}{(x+y)^2} \qquad \ldots(3)$$

Now, $4\left[1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right]=4\left[1-\dfrac{x^2+2xy-y^2}{(x+y)^2}-\dfrac{y^2+2xy-x^2}{(x+y)^2}\right]$

$$=4\left[\frac{(x+y)^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

i.e., $\qquad 4\left[1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right]=4\left[\dfrac{x^2-2xy+y^2}{(x+y)^2}\right]=4\,\dfrac{(x-y)^2}{(x+y)^2} \qquad \ldots(4)$

Thus from (3) and (4) $\left[\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right]^2=4\left[1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right]$

Type - 2 Indirect partial derivatives

This is analogous to the concept of the differentiation of implict functions in the case of ordinary derivatives.

Given a relation of the form $f(x, y, z)=c$, we need to identify the dependent variable and the associated independent variables based on the required partial derivatives. For example if $\dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial x\,\partial y}, \dfrac{\partial^2 z}{\partial y^2}$ etc, are required we have to obviously infer that z is a function of x and y.

Suppose we have $x=f(u, v)$ and $y=g(u, v)$ and let us suppose that $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ is required. In this case we have to express u in terms of x and y as a single equation by eliminating v from the given relations.

It is important to note the following difference

$$\frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)} \text{ and vice - versa, where as}$$

$$\frac{\partial u}{\partial x} \neq \frac{1}{\left(\dfrac{\partial x}{\partial u}\right)} \quad \text{and} \quad \frac{\partial u}{\partial y} \neq \frac{1}{\left(\dfrac{\partial y}{\partial u}\right)} \quad \text{etc.}$$

WORKED PROBLEMS

26. *If* $x^x y^y z^z = c$ *show that* $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$

and hence deduce that $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$ *when* $x = y = z$

>> We have $x^x y^y z^z = c$ and we have to treat z as a function of x and y in order to find the required partial derivative.

Taking logarithms on both sides of the given equation we get

$$x \log x + y \log y + z \log z = \log c$$

Differentiating partially *w.r.t.* y bearing in mind that z is a function of x & y we get

$$0 + \left(y \cdot \frac{1}{y} + 1 \cdot \log y\right) + \left(z \cdot \frac{1}{z} + 1 \cdot \log z\right)\frac{\partial z}{\partial y} = 0$$

$$\therefore \quad \frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}$$

Now $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial}{\partial x}\left\{\frac{(1+\log y)}{(1+\log z)}\right\}$$

ie., $$\frac{\partial^2 z}{\partial x \partial y} = -\left\{\frac{(1+\log z)\cdot 0 - (1+\log y)\cdot \dfrac{1}{z}\dfrac{\partial z}{\partial x}}{(1+\log z)^2}\right\} \qquad \ldots (1)$$

Taking a note that the given function is symmetric we can write,

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \qquad \ldots (2)$$

Using (2) in (1)we get,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

Further when $x = y = z$, the R.H.S of the above expression on replacing y and z by x assumes the form

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x(\log e+\log x)} = -\frac{1}{x\log ex} = -(x\log ex)^{-1}$$

Thus $\left[\dfrac{\partial^2 z}{\partial x\,\partial y}\right]_{x=y=z} = -(x\log ex)^{-1}$

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27. *If* $r^2 = (x-a)^2+(y-b)^2+(z-c)^2$, *show that* $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}+\dfrac{\partial^2 r}{\partial z^2} = \dfrac{2}{r}$

>> $r^2 = (x-a)^2+(y-b)^2+(z-c)^2$ is a symmetric function.

(We do not prefer to write the expression for r)

Differentiating partially *w.r.t.* x on both sides, we have,

$$2r\frac{\partial r}{\partial x} = 2(x-a) \text{ or } \frac{\partial r}{\partial x} = \frac{(x-a)}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{(x-a)}{r}\right]$$

Applying quotient rule we get,

$$\frac{\partial^2 r}{\partial x^2} = \frac{r\cdot 1-(x-a)\cdot\dfrac{\partial r}{\partial x}}{r^2} = \frac{r-[(x-a)\cdot(x-a)/r]}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r^2-(x-a)^2}{r^3} \qquad \ldots(1)$$

Similarly $\dfrac{\partial^2 r}{\partial y^2} = \dfrac{r^2-(y-b)^2}{r^3}$ ...(2)

$$\frac{\partial^2 r}{\partial z^2} = \frac{r^2-(z-c)^2}{r^3} \qquad \ldots(3)$$

Adding the results (1), (2) and (3) we get,

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2} = \frac{1}{r^3}\left\{3r^2-[(x-a)^2+(y-b)^2+(z-c)^2]\right\}$$

$$= \frac{1}{r^3}(3r^2-r^2) = \frac{2r^2}{r^3} = \frac{2}{r}$$

Thus $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}+\dfrac{\partial^2 r}{\partial z^2} = \dfrac{2}{r}$

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28. *If* $x = r\cos\theta$ *and* $y = r\sin\theta$ *prove that*

(a) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left\{\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right\}$ (b) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$

>> Observing the desired result, we need to first express r as a function of x & y

$$x = r\cos\theta,\ y = r\sin\theta \text{ gives } x^2 + y^2 = r^2$$

Now consider $r^2 = x^2 + y^2$ which is a symmetric function

Differentiating partialy *w.r.t* x on both sides we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{Similarly} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Now $\dfrac{\partial^2 r}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial r}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{x}{r}\right)$

Applying quotient rule, we have,

$$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \dfrac{\partial r}{\partial x}}{r^2} = \frac{r - x \cdot \dfrac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3}$$

Similarly $\dfrac{\partial^2 r}{\partial y^2} = \dfrac{r^2 - y^2}{r^3}$

$$\text{L.H.S} = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} = \frac{1}{r^3}\left[2r^2 - (x^2 + y^2)\right] = \frac{2r^2 - r^2}{r^3} = \frac{1}{r}$$

$$\text{Also R.H.S} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\} = \frac{1}{r}\left\{\frac{x^2}{r^2} + \frac{y^2}{r^2}\right\}$$

$$i.e., = \frac{1}{r}\cdot\frac{x^2 + y^2}{r^2} = \frac{1}{r}\cdot\frac{r^2}{r^2} = \frac{1}{r}$$

Thus L.H.S = R.H.S

(b) $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \dfrac{r\cos\theta}{r} = \cos\theta$

Since $x = r\cos\theta$, $\dfrac{\partial x}{\partial r} = \cos\theta$

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

29 *If* $x = r\cos\theta,\ y = r\sin\theta$ *show that*

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial^2}{\partial x^2}(\text{l[illegible]}) \quad \frac{\partial^2}{\partial r^2}\ \text{[illegible]} \quad \frac{\cos 2\theta}{r}$$

>> Observing the required partial derivatives, we need to express r and θ in terms of x, y.

$$x = r\cos\theta, \quad y = r\sin\theta \text{ gives } x^2+y^2 = r^2$$

Also $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x}$ or $\tan\theta = \dfrac{y}{x} \;\Rightarrow\; \theta = \tan^{-1}(y/x)$

(i) Consider $\theta = \tan^{-1}(y/x)$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

Thus $$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \ldots(1)$$

(ii) Let $u = \log r = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$

Now $$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \ldots(2)$$

Since $u = \log\sqrt{x^2+y^2}$ is a symmetric function, we have similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{-(y^2-x^2)}{(x^2+y^2)^2} \quad \ldots(3)$$

Thus from (1), (2) and (3) we have

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial x^2} = \frac{-\partial^2 u}{\partial y^2} \quad \text{where } u = \log r.$$

Since each of these is equal to $\dfrac{y^2 - x^2}{(x^2+y^2)^2}$ we have on substituting $x = r\cos\theta$, $y = r\sin\theta$ the same becomes

$$\frac{r^2(\sin^2\theta - \cos^2\theta)}{(r^2)^2} = \frac{-(\cos^2\theta - \sin^2\theta)}{r^2} = \frac{-\cos 2\theta}{r^2}$$

Thus $\dfrac{\partial^2\theta}{\partial x\,\partial y} = \dfrac{\partial^2}{\partial x^2}(\log r) = -\dfrac{\partial^2}{\partial y^2}(\log r) = -\dfrac{\cos 2\theta}{r^2}$

---

30. *If $u = f(r)$ where $r = \sqrt{x^2+y^2+z^2}$, then show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$$ *Hence deduce the expression if $u = r^n$*

\>> $u = f(r)$ where r is a function of x, y, z.

$\therefore \quad \dfrac{\partial u}{\partial x} = f'(r)\,\dfrac{\partial r}{\partial x}$ ... (1)

But $r = \sqrt{x^2+y^2+z^2}$ gives $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{r}$

Thus $\dfrac{\partial u}{\partial x} = f'(r)\cdot\dfrac{x}{r}$

$\therefore \quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left\{f'(r)\cdot\dfrac{x}{r}\right\}$.

Applying product rule we have,

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{r\cdot 1 - x\dfrac{\partial r}{\partial x}}{r^2}\right] + f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r}$$

$$= \frac{f'(r)}{r^2}\left[r - x\cdot\frac{x}{r}\right] + f''(r)\frac{x}{r}\cdot\frac{x}{r}$$

$$= \frac{f'(r)}{r^2}\cdot\frac{(r^2-x^2)}{r} + f''(r)\cdot\frac{x^2}{r^2}$$

$\therefore \quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{f'(r)}{r^3}(r^2-x^2) + f''(r)\cdot\dfrac{x^2}{r^2}$ ... (2)

Similarly $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{f'(r)}{r^3}(r^2-y^2) + f''(r)\cdot\dfrac{y^2}{r^2}$ ... (3)

$$\frac{\partial^2 u}{\partial z^2} = \frac{f'(r)}{r^3}(r^2 - z^2) + f''(r) \cdot \frac{z^2}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{f'(r)}{r^3}\left\{(r^2 - x^2) + (r^2 - y^2) + (r^2 - z^2)\right\} + \frac{f''(r)}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{f'(r)}{r^3}\left\{3r^2 - (x^2 + y^2 + z^2)\right\} + \frac{f''(r)}{r^2} \cdot r^2 \qquad \because x^2 + y^2 + z^2 = r^2$$

$$= \frac{f'(r)}{r^3} \cdot 2r^2 + f''(r) = \frac{2}{r} f'(r) + f''(r)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r} f'(r)$

Next if $u = r^n$ then $f'(r) = nr^{n-1}$, $f''(r) = n(n-1)r^{n-2}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= n(n-1)r^{n-2} + \frac{2}{r} \cdot n r^{n-1} = n(n-1)r^{n-2} + 2n r^{n-2} = r^{n-2}(n^2 - n + 2n)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = (n^2 + n) r^{n-2} = n(n+1) r^{n-2}$

---

31. *If* $u = f(r)$ *and* $x = r\cos\theta$, $y = r\sin\theta$, *prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

>> Observing the required partial derivatives we conclude that u must be a function of x, y. But $u = f(r)$ by data and hence we need to have r as a function of x, y. Since $x = r\cos\theta$; $y = r\sin\theta$ we have $x^2 + y^2 = r^2$.

$\therefore$ we have $u = f(r)$ where $r = \sqrt{x^2 + y^2}$.

This example is virtually similar to the previous example wherein we had another term z^2. Proceeding on the same lines one can easily obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r^3}(r^2 - x^2) + \frac{f''(r)}{r^2} \cdot x^2 \text{ and}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}(r^2 - y^2) + \frac{f''(r)}{r^2} \cdot y^2$$

Adding these results we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}\left[2r^2 - (x^2 + y^2)\right] + \frac{f''(r)}{r^2}(x^2 + y^2)$$

$$= \frac{f'(r)}{r^3} \cdot r^2 + \frac{f''(r)}{r^2} \cdot r^2 = \frac{1}{r} f'(r) + f''(r)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

---

32. *If* $r^2 = x^2 + y^2 + z^2$ *show that* $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)e^r = \left(1 + \dfrac{2}{r}\right)e^r$

>> If $u = e^r$, then we have to show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \left(1 + \dfrac{2}{r}\right)e^r$

Here $u = e^r$ where $r^2 = x^2 + y^2 + z^2$, is a symmetric function

$\therefore \quad \dfrac{\partial u}{\partial x} = e^r \dfrac{\partial r}{\partial x}$

But $r^2 = x^2 + y^2 + z^2$ and hence $2r \dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Thus $\dfrac{\partial u}{\partial x} = e^r \cdot \dfrac{x}{r}$

Now $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(e^r \cdot \dfrac{x}{r}\right)$

i.e., $= e^r\left[\dfrac{r \cdot 1 - x\dfrac{\partial r}{\partial x}}{r^2}\right] + e^r \dfrac{\partial r}{\partial x} \cdot \dfrac{x}{r}$

$= \dfrac{e^r}{r^2}\left[r - x \cdot \dfrac{x}{r}\right] + e^r \cdot \dfrac{x}{r} \cdot \dfrac{x}{r}$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{e^r}{r^3}(r^2 - x^2) + \dfrac{e^r}{r^2} x^2 \quad \ldots (1)$

Similarly $$\frac{\partial^2 u}{\partial y^2} = \frac{e^r}{r^3}(r^2 - y^2) + \frac{e^r}{r^2}\cdot y^2 \qquad \ldots (2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{e^r}{r^3}(r^2 - z^2) + \frac{e^r}{r^2}\cdot z^2 \qquad \ldots (3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{e^r}{r^3}\left[3r^2 - (x^2 + y^2 + z^2)\right] + \frac{e^r}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{e^r}{r^3}\cdot 2r^2 + \frac{e^r}{r^2}\cdot r^2 = e^r\cdot\frac{2}{r} + e^r$$

Thus $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = e^r\left(\frac{2}{r} + 1\right)$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

>> We need to treat u as a function of x, y, z.

Consider $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$

Differentiating partially w.r.t. x we get

$$\left\{x^2\cdot -\frac{1}{(a^2+u)^2}u_x + 2x\cdot\frac{1}{a^2+u}\right\} + \left\{y^2\cdot -\frac{1}{(b^2+u)^2}u_x\right\}$$
$$+\left\{z^2\cdot -\frac{1}{(c^2+u)^2}u_x\right\} = 0$$

or $$u_x\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{2x}{a^2+u}$$

We get similar equations related to u_y and u_z with the repetition of the expression in the square bracket [ ].

Let $v = \dfrac{x^2}{(a^2+u)^2} + \dfrac{y^2}{(b^2+u)^2} + \dfrac{z^2}{(c^2+u)^2}$ for convenience.

Now we have three equations :

$$u_x\cdot v = \frac{2x}{a^2+u}\,,\quad u_y\cdot v = \frac{2y}{b^2+u}\,,\quad u_z\cdot v = \frac{2z}{c^2+u} \qquad \ldots (1)$$

Squaring and adding these we get,

$$(u_x^2+u_y^2+u_z^2)\,v^2 = 4\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right] = 4v$$

$$\therefore \quad \text{L.H.S} = u_x^2+u_y^2+u_z^2 = \frac{4v}{v^2} = \frac{4}{v} \quad \ldots (2)$$

Taking a note of the required expression in R.H.S, let us multiply the three equations present in (1) respectively by $x,\ y,\ z$ and add.

$$\therefore \quad x\,u_x\,v+y\,u_y\,v+z\,u_z\,v = 2\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

or $$2\,(x\,u_x+y\,u_y+z\,u_z) = \frac{4}{v}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

But $$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u} = 1, \text{ by data.}$$

Hence $$2\,(x\,u_x+y\,u_y+z\,u_z) = \frac{4}{v} \quad \ldots (3)$$

$\therefore$ from (2) and (3), $u_x^2+u_y^2+u_z^2 = 2\,(x\,u_x+y\,u_y+z\,u_z)$

3.13 | Total differentiation

If $u = f(x,\ y)$ then the total differential or the exact differential of u is defined as

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \quad \ldots (1)$$

Differentiation of composite and implicit functions

If $u = f(x,\ y)$ where x and y are functions of the independent variable t then u *is said to be a composite function of the single variable* t.

Also if $u = f(x,\ y)$ where both x and y are functions of two independent variables $r,\ s$ then u *is said to be a composite function of the two variables* r *and* s

The principle of differentiation of composite function is very much similar to that of the function of a function rule associated with the ordinary derivative of a function of a single independent variable.

We discuss two types involving partial derivatives.

Type - (i). Total derivative rule

If $u = f(x,\ y)$ where $x = x(t)$ and $y = y(t)$ then u is a composite function of the single variable t. Therefore in principle we should be able to differentiate u with respect to t which is an ordinary derivative.

Thus we have with reference to (1),

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad \ldots (2)$$

This is called as the total derivative of u.

Type - (ii). Chain rule

If $u = f(x, y)$ where $x = x(r, s)$ and $y = y(r, s)$ then u is a composite function of two independent variables r, s. Therefore in principle we should we able to differentiate u w.r.t. r and also w.r.t. s partially. Thus we have the following chain rules for the two partial derivatives. It is convenient to write the rule having the data analysed in the following format.

$$\begin{tikzcd} & & & \frac{\partial u}{\partial r} \\ u \to (x, y) \to (r, s) \Rightarrow u \to (r, s) \arrow[urrr] \arrow[drrr] & & & \\ & & & \frac{\partial u}{\partial s} \end{tikzcd}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \; ; \; \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \qquad \ldots (3)$$

Note - 1. *The rules (2) and (3) can be established from the basic limit form definition of a partial derivative.*

2 . *The rules (2) and (3) can be extended to functions involving more than two independent variables.*

3 . *The rules (2) and (3) can be successively applied for getting higher order derivatives of the given function.*

4. *The symbol → is used as a to indicate the composition of the variables so that the associated rule can be written conveniently.*

Corollary : Differentiation of implicit functions

Let $u = f(x, y)$ and let y be a function of x and also $f(x, y) = c$, c being a constant.

ie., $u = f(x, y)$ where $y = y(x)$. Hence by the rule of the total derivative,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

ie., $$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

Since $u = f(x, y) = c$ we have $\dfrac{du}{dx} = 0$ and the above equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = -\frac{u_x}{u_y}$$

Thus we can say that if $u = f(x, y) = c$ then

$$\frac{dy}{dx} = -\frac{u_x}{u_y} \quad \text{or} \quad \frac{dy}{dx} = -\frac{f_x}{f_y}$$

Remark : *It is a known fact that a function $f(x, y) = c$ is called an implicit function and we are conversant in finding the derivative of an implicit function. Here we have a formula for $\frac{dy}{dx}$ in terms of partial derivatives and the same can be successively applied for higher order derivatives.*

WORKED PROBLEMS

Working procedure for problems

- We have to analyse the composition of the variables and write the appropriate formula.
- We then substitute for the possible derivatives in the formula and simplify according to the requirement of the desired result.
- Sometimes we may have to change the composition of the variables so as to achieve the desired result.

Find the total differential of the following functions.

34. We have $u = x^3 + xy^2 + x^2y + y^3$; $u \to (x, y)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus $\quad du = (3x^2 + y^2 + 2xy)\,dx + (2xy + x^2 + 3y^2)\,dy$

35. We have $x = r\sin\theta\cos\phi$; $x \to (r, \theta, \phi)$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi$$

Thus $\quad dx = (\sin\theta\cos\phi)\,dr + (r\cos\theta\cos\phi)\,d\theta + (-r\sin\theta\sin\phi)\,d\phi$

36. We have $\psi = u_1 u_2^2 + u_2 u_3^2 + u_3 u_1^2$; $\psi \to (u_1, u_2, u_3)$

$$\therefore \quad d\psi = \frac{\partial \psi}{\partial u_1} du_1 + \frac{\partial \psi}{\partial u_2} du_2 + \frac{\partial \psi}{\partial u_3} du_3$$

Thus $\quad d\psi = (u_2^2 + 2u_1 u_3)\, du_1 + (u_3^2 + 2u_1 u_2)\, du_2 + (u_1^2 + 2u_2 u_3)\, du_3$

37. We have $\phi = xy^2 z^3$; $\phi \to (x, y, z)$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

Thus $\quad d\phi = (y^2 z^3)\, dx + (2xyz^3)\, dy + (3xy^2 z^2)\, dz$

[illegible] by direct substitution

38. $\quad z = xy^2 + x^2 y$; $x = at,\ y = 2at$

$z \to (x, y) \to t \quad \Rightarrow z \to t$ & $\dfrac{dz}{dt}$ is the total derivative.

$$\therefore \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)\, a + (2xy + x^2)\, 2a$$

$$= (4a^2t^2 + 4a^2t^2)\, a + (4a^2t^2 + a^2t^2)\, 2a$$

$$= (8a^2t^2)\, a + (5a^2t^2)\, 2a = 8a^3t^2 + 10a^3t^2 = 18a^3t^2$$

$\therefore$ the total derivative $\dfrac{dz}{dt} = 18a^3t^2$ . . . (1)

Now by direct substitution we have,

$$z = xy^2 + x^2y = (at)(2at)^2 + (at)^2(2at) = 4a^3t^3 + 2a^3t^3$$

ie., $\quad z = 6a^3t^3$

Now differentiating *w.r.t* t,

$$\frac{dz}{dt} = 6a^3 \cdot 3t^2 = 18a^3t^2$$

From (1) & (2) the **result is verified.**

39. $u = x^2 + y^2 - z^2, \; x = e^t, \; y = e^t \cosh t, \; z = e^t \sinh t$

$u \to (x, y, z) \to t \} \Rightarrow u \to t$ & $\dfrac{du}{dt}$ is the total derivative.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (2x)(e^t) + (2y)(e^t \sinh t + e^t \cosh t)$$

$$- (2z)(e^t \cosh t + e^t \sinh t)$$

Substituting for x, y, z, we get

$$2e^t \cdot e^t + e^t \cosh t (\sinh t + \cosh t) - e^t \sinh t (\cosh t + \sinh t)$$

$$2e^t \cdot e^t \{1 + \cosh t \sinh t + \cosh^2 t - \sinh t \cosh t - \sinh^2 t$$

$$= 2e^{2t} \{1 + (\cosh^2 t - \sinh^2 t)\}$$

$$= 2e^{2t}(1+1) = 4e^{2t}$$

$\therefore$ the total derivative $\dfrac{du}{dt} = 4e^{2t}$ ... (1)

Now by direct substitution we have,

$$u = x^2 + y^2 - z^2 = (e^t)^2 + (e^t \cosh t)^2 - (e^t \sinh t)^2$$

$$= e^{2t} \{1 + (\cosh^2 t - \sinh^2 t)\} = e^{2t}(1+1) = 2e^{2t}$$

ie., $u = 2e^{2t}$

Differentiating $w.r.t$ t we get,

$\dfrac{du}{dt} = 2(2e^{2t}) = 4e^{2t}$ ... (2)

From (1) and (2) the result is verified.

40. $u = xy + yz + zx \; ; \; x = t\cos t, \; y = t \sin t, \; z = t$

$u \to (x, y, z) \to t \} \Rightarrow u \to t$ & $\dfrac{du}{dt}$ is the total derivative.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (y+z)(-t\sin t + \cos t) + (x+z)(t\cos t + \sin t) + (y+x)(1)$$

$$= (t\sin t + t)(-t\sin t + \cos t) + (t\cos t + t)(t\cos t + \sin t)$$

$$+ (t\sin t + t\cos t)$$

$$= (-t^2 \sin^2 t - t^2 \sin t + t \sin t \cos t + t \cos t)$$
$$+ (t^2 \cos^2 t + t^2 \cos t + t \sin t \cos t + t \sin t) + (t \sin t + t \cos t)$$

$$= t^2 (\cos^2 t - \sin^2 t) + t^2 (\cos t - \sin t)$$
$$+ 2t \sin t \cos t + 2t (\cos t + \sin t)$$

Now at $t = \pi/4$ we have $\cos t = \sin t = 1/\sqrt{2}$

$$\therefore \quad \left(\frac{du}{dt}\right)_{t = \pi/4} = 0 + 0 + 2 \cdot \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{\pi}{4} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

i.e., $$\left(\frac{du}{dt}\right)_{t = \pi/4} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}} \qquad \ldots (1)$$

Now by direct substitution we have,

$$u = xy + yz + zx = t^2 (\sin t \cos t + \sin t + \cos t)$$

i.e., $$u = t^2 (1/2 \cdot \sin 2t + \sin t + \cos t)$$

Differentiating *w.r.t* t, we have,

$$\frac{du}{dt} = t^2 (\cos 2t + \cos t - \sin t) + 2t (1/2 \cdot \sin 2t + \sin t + \cos t)$$

$$\left(\frac{du}{dt}\right)_{t = \pi/4} = \frac{\pi^2}{16} \left(0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{\pi}{2} \left(\frac{1}{2} \cdot 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2} \left(\frac{1}{2} + \frac{2}{\sqrt{2}}\right) = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

$$\therefore \quad \left(\frac{du}{dt}\right)_{t = \pi/4} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}} \qquad \ldots (2)$$

From (1) & (2) the **result is verified.**

11. *If* $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$ *show that* $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$ *and verify the result by direct differentiation.*

$>>$ $$u = \sin^{-1}(x - y) \; ; \; x = 3t, \; y = 4t^3$$

$$u \to (x, y) \to t \,|\, \Rightarrow u \to t \quad \therefore \frac{du}{dt} \text{ is the total derivative.}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} \cdot 3 - \frac{1}{\sqrt{1-(x-y)^2}} (12t^2)$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}} = \frac{3}{\sqrt{\dfrac{1-9t^2+24t^4-16t^6}{(1-4t^2)^2}}}$$

Dividing $(1-9t^2+24t^4-16t^6)$ by $(1-8t^2+16t^4)$ we get $(1-t^2)$.

Thus $\quad \dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$ ... (1)

Now consider $u = \sin^{-1}(x-y) = \sin^{-1}(3t-4t^3)$

In order to differentiate u w.r.t. 't' directly it is convenient by using the substitution $t = \sin\theta$ so that we have,

$$u = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin^{-1}(\sin 3\theta) = 3\theta$$

Thus $\quad u = 3\theta \quad$ and hence $\dfrac{du}{dt} = 3\dfrac{d\theta}{dt} = 3\dfrac{d}{dt}(\sin^{-1} t) = \dfrac{3}{\sqrt{1-t^2}}$

Thus $\quad \dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$ (2)

From (1) and (2) **the result is verified.**

Find $\dfrac{dy}{dx}$ [illegible]

42. $x^{2/3} + y^{2/3} = a^{2/3}$ 43. $x^y + y^x = a$

42. Let $u = f(x, y) = x^{2/3} + y^{2/3} - a^{2/3}$

$$\therefore \quad u_x = \frac{\partial u}{\partial x} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{2}{3}y^{2/3-1} = \frac{2}{3}y^{-1/3}$$

We have $\dfrac{dy}{dx} = \dfrac{-u_x}{u_y} = -\dfrac{2/3\, x^{-1/3}}{2/3\, y^{-1/3}} = -\dfrac{x^{-1/3}}{y^{-1/3}}$

Thus $\quad \dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$

13. Let $u = f(x, y) = a^x + a^y - a^{x+y}$

$\therefore$ $$u_x = \frac{\partial u}{\partial x} = a^x \log a - a^{(x+y)} \log a \cdot 1$$

$$u_y = \frac{\partial u}{\partial y} = a^y \log a - a^{x+y} \log a \cdot 1$$

We have $$\frac{dy}{dx} = -\frac{u_x}{u_y} = -\frac{\left[a^x \log a - a^{x+y} \log a \right]}{\left[a^y \log a - a^{x+y} \log a \right]}$$

$$= -\frac{\log a \, [a^x - a^{x+y}]}{\log a \, [a^y - a^{x+y}]} = -\frac{[a^x - a^{x+y}]}{[a^y - a^{x+y}]}$$

Since $a^x + a^y = a^{x+y}$ by data, we get

$$\frac{dy}{dx} = -\frac{(-a^y)}{-a^x} \quad \text{or} \quad \boldsymbol{\frac{dy}{dx} = -a^{y-x}}$$

14. *If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$ by using partial derivat*

>> Let $u = f(x, y) = x^m y^n - (x+y)^{m+n}$

$\therefore$ $$u_x = \frac{\partial u}{\partial x} = (m x^{m-1}) y^n - (m+n)(x+y)^{m+n-1}$$

$$u_y = \frac{\partial u}{\partial y} = x^m (n y^{n-1}) - (m+n)(x+y)^{m+n-1}$$

Multiplying these partial derivatives by $(x+y)$ we get

$$(x+y) u_x = m x^{m-1} y^n (x+y) - (m+n)(x+y)^{m+n}$$

and $$(x+y) u_y = x^m (n y^{n-1})(x+y) - (m+n)(x+y)^{m+n}$$

ie., $$(x+y) u_x = m x^{m-1} y^n (x+y) - (m+n) x^m y^n$$

and $$(x+y) u_y = x^m (n y^{n-1})(x+y) - (m+n) x^m y^n$$

ie., $$(x+y) u_x = x^{m-1} y^n \left[m(x+y) - (m+n) x \right]$$

and $$(x+y) u_y = x^m y^{n-1} \left[n(x+y) - (m+n) y \right]$$

ie., $$(x+y) u_x = x^{m-1} y^n (my - nx) \qquad \ldots (1)$$

$$(x+y)\,u_y = x^m y^{n-1}(nx-my) \qquad \ldots (2)$$

But $\dfrac{dy}{dx} = -\dfrac{u_x}{u_y} = -\dfrac{(x+y)\,u_x}{(x+y)\,u_y}$.

Using (1) and (2) we get,

$$\frac{dy}{dx} = \frac{-x^{m-1}y^n(my-nx)}{x^m y^{n-1}(nx-my)} = \frac{x^{m-1}y^n(nx-my)}{x^m y^{n-1}(nx-my)}$$

Thus $\dfrac{dy}{dx} = \dfrac{y}{x}$ which is the desired result.

---

45. *Find* $\dfrac{du}{dx}$ *using the concept of composite functions given that* $u = \tan^{-1}(x/y)$ *when* $x^2+y^2 = a^2$

>> Here $[\,u \to (x,\ y)$ and $y \to x\,] \Rightarrow u \to x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial y}\,\frac{dy}{dx} \qquad \ldots (1)$$

Consider $u = \tan^{-1}(x/y)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+(x/y)^2}\cdot\frac{1}{y} = \frac{y^2}{x^2+y^2}\cdot\frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+(x/y)^2}\cdot -\frac{x}{y^2} = \frac{y^2}{x^2+y^2}\cdot -\frac{x}{y^2} = -\frac{x}{x^2+y^2}$$

Also consider $x^2+y^2 = a^2$ and differentiating *w.r.t* x, we get

$$2x+2y\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{x}{y}$$

Using these results in (1) we have,

$$\frac{du}{dx} = \frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\cdot\left(-\frac{x}{y}\right) = \frac{y^2+x^2}{(x^2+y^2)\,y} = \frac{1}{y}$$

Thus $\dfrac{du}{dx} = \dfrac{1}{y} = \dfrac{1}{\sqrt{a^2-x^2}}$

---

46. *If* $u = \sqrt{x^2 + y^2}$ *and* $x^3 + y^3 + 3axy = 5a^2$ *find* $\dfrac{du}{dx}$ when $x = y = a$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\,\frac{dy}{dx} \qquad \ldots (1)$$

We have $u = \sqrt{x^2+y^2}$ $\quad\therefore\quad \dfrac{\partial u}{\partial x} = \dfrac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \dfrac{x}{\sqrt{x^2+y^2}}$

Also $\quad \dfrac{\partial u}{\partial y} = \dfrac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \dfrac{y}{\sqrt{x^2+y^2}}$

Consider $x^3 + y^3 + 3axy = 5a^2$ and differentiate *w.r.t* x

$\therefore \qquad 3x^2 + 3y^2\dfrac{dy}{dx} + 3a\left(x\dfrac{dy}{dx} + y\right) = 0$

or $\qquad \dfrac{dy}{dx} = -\dfrac{(x^2+ay)}{(y^2+ax)}$

Now when $x = y = a$, $\dfrac{\partial u}{\partial x} = \dfrac{a}{\sqrt{2a^2}} = \dfrac{1}{\sqrt{2}}$, $\dfrac{\partial u}{\partial y} = \dfrac{1}{\sqrt{2}}$ and $\dfrac{dy}{dx} = -1.$

Substituting these values in (1) we get

$$\frac{du}{dx} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(-1) = 0$$

Thus when $\quad x = y = a,\ \dfrac{du}{dx} = 0$

---

47. *If* $u = e^{ax+by} f(ax-by)$, *prove that* $b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y} = 2abu$ *by using the concept of composite functions.*

\>\> We have $u = e^{ax+by} f(ax-by)$

Let $\quad r = ax+by,\ s = ax-by$ so that $u = e^r f(s)$

Hence $\quad \{u \to (r, s) \to (x, y)\} \Rightarrow u \to x, y$

We have chain rules,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}\ ;\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$$

ie., $\qquad \dfrac{\partial u}{\partial x} = e^r f(s)\,a + e^r f'(s)\,a\ ;\ \dfrac{\partial u}{\partial y} = e^r f(s)\,b + e^r f'(s)(-b)$

ie., $\qquad \dfrac{\partial u}{\partial x} = a\,e^r\left[f(s)+f'(s)\right]\ ;\ \dfrac{\partial u}{\partial y} = b\,e^r\left[f(s)-f'(s)\right]$

Now consider $b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y}$

$$= b \cdot ae^r\left[f(s)+f'(s)\right]+a \cdot be^r\left[f(s)-f'(s)\right]$$

$$= abe^r f(s)+abe^r f'(s)+abe^r f(s)-abe^r f'(s)$$

$$= 2\,ab\,e^r f(s) = 2\,a\,b\,u$$

Thus $\quad \boldsymbol{b\dfrac{\partial u}{\partial x} + a\dfrac{\partial u}{\partial y} = 2\,a\,b\,u}$

Remark : *Refering to Problem - 17, it may be observed that the same problem has been worked out directly.*

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48. *If* $u = f\left(\dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

>> Here we need to convert the given function u into a composite function.

Let $\quad u = f(p, q, r)$ where $p = \dfrac{x}{y}$, $q = \dfrac{y}{z}$, $r = \dfrac{z}{x}$

ie., $\quad \{u \to (p, q, r) \to (x, y, z)\} \Rightarrow u \to x, y, z$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\text{ie.,} \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{1}{y} + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}\cdot\left(-\frac{z}{x^2}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} \qquad \ldots(1)$$

Similarly by symmetry we can write,

$$y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} \qquad \ldots(2)$$

$$z\frac{\partial u}{\partial z} = \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q} \qquad \ldots(3)$$

Adding (1), (2) and (3) we get $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

---

49. *If* $u = f(x-y,\ y-z,\ z-x)$ *show that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} - 0$

>> *(This example is similar to the previous example)*

Let $\quad u = f(p, q, r)$ where $p = x-y,\ q = y-z,\ r = z-x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

ie., $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot 1 + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}(-1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \qquad \ldots (1)$$

Similarly we have by symmetry,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \qquad \ldots (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \qquad \ldots (3)$$

Adding (1), (2) and (3) we get, $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

---

50. *If* $z = f(x, y)$ *and* $x = u - v$, $y = uv$ *show that*

(i) $(u+v)\dfrac{\partial z}{\partial x} = u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v}$ (ii) $(u+v)\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$

>> $\{z \to (x, y) \to (u, v)\} \Rightarrow z \to u, v$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

ie., $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot 1 + \frac{\partial z}{\partial y}\cdot v = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} \qquad \ldots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot(-1) + \frac{\partial z}{\partial y}\cdot u = -\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \qquad \ldots (2)$$

(i) Consider R.H.S $= u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v}$

$$= u\left[\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right] - v\left[\frac{-\partial z}{\partial x} + u\frac{\partial z}{\partial y}\right]$$

$= (u+v)\dfrac{\partial z}{\partial x} = \text{L.H.S}$ $\quad \therefore$ **R.H.S = L.H.S**

(ii) Consider R.H.S $= \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$

Adding (1) and (2) we have,

$\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} = (u+v)\dfrac{\partial z}{\partial y} = \text{L.H.S}$ $\quad \therefore$ **R.H.S = L.H.S**

---

51. *If $z = x^2 + y^2$ where $x = e^u \sin v$, $y = e^u \cos v$ find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ as a composite function and verify the results by direct substitution.*

>> $\{z \to (x, y) \to (u, v)\} \Rightarrow z \to (u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\therefore \quad \frac{\partial z}{\partial u} = (2x)(e^u \sin v) + (2y)(e^u \cos v)$$

$$= 2e^u \left\{ e^u \sin v (\sin v) + e^u \cos v (\cos v) \right\}$$

$$= 2e^u \cdot e^u (\sin^2 v + \cos^2 v) = 2e^{2u}$$

$$\frac{\partial z}{\partial v} = (2x)(e^u \cos v) + (2y)(-e^u \sin v)$$

$$= 2e^u \left\{ e^u \sin v (\cos v) - e^u \cos v (\sin v) \right\} = 0$$

Thus as a composite function we have obtained

$$\frac{\partial z}{\partial u} = 2e^{2u} \; ; \; \frac{\partial z}{\partial v} = 0 \qquad \ldots (1)$$

Now consider $z = x^2 + y^2$ by direct substitution.
$z = (e^u \sin v)^2 + (e^u \cos v)^2 = e^{2u}(\sin^2 v + \cos^2 v) = e^{2u}$

Differentiating partially *w.r.t* u and also *w.r.t* v we get

$$\frac{\partial z}{\partial u} = 2e^{2u} \text{ and } \frac{\partial z}{\partial v} = 0 \qquad \ldots (2)$$

From (1) and (2) we conclude that **the result is verified.**

---

52. *If* [illegible]

(i) [illegible]

(ii) [illegible]

>> (i) $z \to \{(x, y) \to (u, v)\} \Rightarrow z \to (u, v)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

ie., $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v) \quad \ldots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(-e^u \sin v) \quad \ldots (2)$$

Squaring and adding (1), (2) and collecting terms suitably we have,

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left(\frac{\partial z}{\partial x}\right)^2 [\sin^2 v + \cos^2 v] + e^{2u}\left(\frac{\partial z}{\partial y}\right)^2 [\cos^2 v + \sin^2 v]$$

$$+ 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} e^{2u} \sin v \cos v - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} e^{2u} \cos v \sin v$$

Thus $$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}$$

(ii) Consider R.H.S $= e^{-u} \sin v \dfrac{\partial z}{\partial u} + e^{-u} \cos v \dfrac{\partial z}{\partial v}$

$$= e^{-u} \sin v \left\{\frac{\partial z}{\partial x} e^u \sin v + \frac{\partial z}{\partial y} e^u \cos v\right\} + e^{-u} \cos v \left\{\frac{\partial z}{\partial x} e^u \cos v - \frac{\partial z}{\partial y} e^u \sin v\right\}$$

$$= \frac{\partial z}{\partial x} \sin^2 v + \frac{\partial z}{\partial y} \sin v \cos v + \frac{\partial z}{\partial x} \cos^2 v - \frac{\partial z}{\partial y} \cos v \sin v$$

$$= \frac{\partial z}{\partial x}(\sin^2 v + \cos^2 v) = \frac{\partial z}{\partial x} = \text{L.H.S}$$

Thus **R.H.S = L.H.S**

Aliter . The problem can also be done by changing the independent variables x, y to u, v by elimination.

$$x = e^u \sin v, \; y = e^u \cos v.$$

Squaring and adding : $x^2 + y^2 = e^{2u}$ or $\log(x^2 + y^2) = 2u$

Dividing, we have $\dfrac{x}{y} = \tan v$ or $v = \tan^{-1}(x/y)$

We can now write $z = f(u, v)$ where

$$u = \frac{1}{2}\log(x^2 + y^2), \; v = \tan^{-1}(x/y)$$

ie., $\{z \to (u, v) \to (x, y)\} \Rightarrow z \to x, y$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

But $\quad \dfrac{\partial u}{\partial x} = \dfrac{1}{2} \cdot \dfrac{2x}{x^2+y^2} = \dfrac{e^u \sin v}{e^{2u}} = e^{-u}\sin v$

$$\frac{\partial v}{\partial x} = \frac{1}{1+(x/y)^2} \cdot \frac{1}{y} = \frac{y^2}{x^2+y^2} \cdot \frac{1}{y}$$

$$= \frac{y}{x^2+y^2} = \frac{e^u \cos v}{e^{2u}} = e^{-u}\cos v$$

Thus $\quad \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial u}(e^{-u}\sin v) + \dfrac{\partial z}{\partial v}(e^{-u}\cos v)$

or $\quad \dfrac{\partial z}{\partial x} = e^{-u}\left(\sin v \dfrac{\partial z}{\partial u} + \cos v \dfrac{\partial z}{\partial v}\right)$

Thus **L.H.S = R.H.S**

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53. *If* [illegible] $f(x, y)$ *where* x [illegible] $r\cos\theta$ *and* y [illegible] $r\sin\theta$

[illegible]

\>> $\quad \{z \to (x, y) \to (r, \theta)\} \Rightarrow z \to (r, \theta)$

$$\therefore \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\ ; \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

ie., $\quad \dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta \qquad \ldots (1)$

and $\quad \dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}(r\cos\theta) = r\left[\dfrac{-\partial z}{\partial x}\sin\theta + \dfrac{\partial z}{\partial y}\cos\theta\right]$

or $\quad \dfrac{1}{r}\dfrac{\partial z}{\partial \theta} = \dfrac{-\partial z}{\partial x}\sin\theta + \dfrac{\partial z}{\partial y}\cos\theta \qquad \ldots (2)$

Squaring and adding (1), (2) and collecting suitable terms we have,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2[\cos^2\theta + \sin^2\theta] + \left(\frac{\partial z}{\partial y}\right)^2[\sin^2\theta + \cos^2\theta]$$
$$+ 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

$\therefore \quad \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$ *ie*, **R.H.S = L.H.S**

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54. *If* $u(x, y)$ *and* $v(x, y)$ *are such that* $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ *and* $\dfrac{\partial v}{\partial x} = \dfrac{-\partial u}{\partial y}$ *and*

$x = r\cos\theta,\ y = r\sin\theta$ *prove that* $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta},\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

>> $\{(u, v) \to (x, y) \to (r, \theta)\} \Rightarrow (u, v) \to (r, \theta)$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\ ;\ \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\ ;\ \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

ie., $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad \ldots(1)$$

Next, by using the data we have,

$$\frac{\partial v}{\partial r} = \frac{-\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta \qquad \ldots(2)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

Next by using the data we have,

$$\frac{\partial v}{\partial \theta} = \frac{-\partial u}{\partial y}(-r\sin\theta) + \frac{\partial u}{\partial x}(r\cos\theta),$$

$$\therefore \quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\sin\theta - \frac{\partial u}{\partial y}\cos\theta \qquad \ldots(3)$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \qquad \ldots(4)$$

By comparing that (1) & (4) and (2) & (3), the desired result follows,

Thus $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta},\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

---

55. *If* $z = f(x, y)$ *where* $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

prove that $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$

>> $\{z \to (x, y) \to (u, v)\} \Rightarrow z \to (u, v)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\ ;\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

ie., $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^{u} + \frac{\partial z}{\partial y}(-e^{-u}) \qquad \ldots (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \qquad \ldots (2)$$

Consider R.H.S $= \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$ and $(1)-(2)$ yields

$$\frac{\partial z}{\partial x}(e^{u}+e^{-v}) - \frac{\partial z}{\partial y}(e^{-u}-e^{v}) = \frac{\partial z}{\partial x} \cdot x - \frac{\partial z}{\partial y} \cdot y$$

Thus $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$ *ie.,* **R.H.S = L.H.S**

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56. If $u = x^{n}\, g(y/x,\, z/x)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = nu$

>> $$u = x^{n} g\left(y/x,\, z/x\right) = x^{n} g(p,\, q) \qquad \ldots (1)$$

where $p = y/x$ and $q = z/x$

Now $$\frac{\partial u}{\partial x} = x^{n}\frac{\partial}{\partial x}\Big[g(p,\, q)\Big] + n\,x^{n-1} g(p,\, q)$$

$$\frac{\partial u}{\partial y} = x^{n}\frac{\partial}{\partial y}\Big[g(p,\, q)\Big]$$

$$\frac{\partial u}{\partial z} = x^{n}\frac{\partial}{\partial z}\Big[g(p,\, q)\Big]$$

We need to apply chain rule for computing the partial derivatives present in the R.H.S of the three equations.

$$\frac{\partial}{\partial x}\Big[g(p,\, q)\Big] = \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial g}{\partial p}\left(-\frac{y}{x^2}\right) + \frac{\partial g}{\partial q}\left(-\frac{z}{x^2}\right)$$

$$\frac{\partial}{\partial y}\Big[g(p,\, q)\Big] = \frac{\partial g}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial y} = \frac{\partial g}{\partial p} \cdot \frac{1}{x} + \frac{\partial g}{\partial q} \cdot 0$$

$$\frac{\partial}{\partial z}\Big[g(p,\, q)\Big] = \frac{\partial g}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial z} = \frac{\partial g}{\partial p} \cdot 0 + \frac{\partial g}{\partial q} \cdot \frac{1}{x}$$

Hence, $$\frac{\partial u}{\partial x} = x^{n}\left[-\frac{y}{x^2}\frac{\partial g}{\partial p} - \frac{z}{x^2}\frac{\partial g}{\partial q}\right] + n\,x^{n-1} g(p,\, q)$$

or $$\frac{\partial u}{\partial x} = -x^{n-2}\left[y\frac{\partial g}{\partial p} + z\frac{\partial g}{\partial q}\right] + n\,x^{n-1} g(p,\, q) \qquad (2)$$

$$\frac{\partial u}{\partial y} = x^n \left[\frac{\partial g}{\partial p} \cdot \frac{1}{x} \right] = x^{n-1} \frac{\partial g}{\partial p} \qquad \dots (3)$$

$$\frac{\partial u}{\partial z} = x^n \left[\frac{\partial g}{\partial q} \cdot \frac{1}{x} \right] = x^{n-1} \frac{\partial g}{\partial q} \qquad \dots (4)$$

Now $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z}$ as a consequence of (2), (3) and (4)

$$= x \left\{ -x^{n-2} \left[y \frac{\partial g}{\partial p} + z \frac{\partial g}{\partial q} \right] + n\, x^{n-1} g(p, q) \right\} + y \cdot x^{n-1} \frac{\partial g}{\partial p} + z \cdot x^{n-1} \frac{\partial g}{\partial q}$$

$$= -x^{n-1} y \frac{\partial g}{\partial p} - x^{n-1} z \frac{\partial g}{\partial q} + n x^n g(p, q) + x^{n-1} y \frac{\partial g}{\partial p} + x^{n-1} z \frac{\partial g}{\partial q}$$

$$= n \left[x^n g(p, q) \right] = n\, u, \text{ by using (1).}$$

Thus $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = n\, u$, as required.

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57. If $v = x + ct$, $w = x - ct$ and $u = f(v, w)$

show that $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2} \dfrac{\partial^2 u}{\partial t^2} = 4 \dfrac{\partial^2 u}{\partial v\, \partial w}$

>> $\quad |u \to (v, w) \to (x, t)| \Rightarrow \quad u \to (x, t)$

We have $v = x + ct$, $w = x - ct$, by data.

By chain rule, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial v} \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial w} \dfrac{\partial w}{\partial x}$

ie., $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial v} \cdot 1 + \dfrac{\partial u}{\partial w} \cdot 1$ or $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial v} + \dfrac{\partial u}{\partial w} = g\ (say)$

Next $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x} \left(\dfrac{\partial u}{\partial x} \right) = \dfrac{\partial g}{\partial x}$ and $g \to (v, w)$

Again by chain rule $\dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial v} \dfrac{\partial v}{\partial x} + \dfrac{\partial g}{\partial w} \dfrac{\partial w}{\partial x}$

ie., $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial v} \left(\dfrac{\partial u}{\partial v} + \dfrac{\partial u}{\partial w} \right) \cdot 1 + \dfrac{\partial}{\partial w} \left(\dfrac{\partial u}{\partial v} + \dfrac{\partial u}{\partial w} \right) \cdot 1$

ie., $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v^2} + 2 \frac{\partial^2 u}{\partial v\, \partial w} + \frac{\partial^2 u}{\partial w^2} \qquad \dots (1)$$

Now $\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial v} \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial w} \dfrac{\partial w}{\partial t}$

ie., $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \cdot c + \frac{\partial u}{\partial w} \cdot (-c) = c\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right] = h \text{ (say)}$$

Next $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial h}{\partial t}$$

Again by chain rule, $$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial h}{\partial w}\frac{\partial w}{\partial t}$$

ie, $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial v}\left\{c\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right]\right\} c + \frac{\partial}{\partial w}\left\{c\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right]\right\} \cdot (-c)$$

ie, $$\frac{\partial^2 u}{\partial t^2} = c^2\left\{\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v\,\partial w} + \frac{\partial^2 u}{\partial w^2}\right\}$$

or $$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v\,\partial w} + \frac{\partial^2 u}{\partial w^2} \qquad \ldots (2)$$

Thus (1) – (2) will give us,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial v\,\partial w} \quad \text{as required}$$

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58. *If* $u = e^x \sin y,\ v = e^x \cos y$ *and* $w = f(u, v)$, *prove that*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right)$$

\>> $\{w \to (u, v) \to (x, y)\} \Rightarrow \qquad w \to x, y$

$u = e^x \sin y,\ v = e^x \cos y$

By chain rule, $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

ie., $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}(e^x \sin y) + \frac{\partial w}{\partial v}(e^x \cos y)$$

or $$\frac{\partial w}{\partial x} = u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} = g \text{ (say)}$$

Next $$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial g}{\partial x} \quad \text{and } g \to u, v$$

Again by chain rule, $$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

i.e., $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial}{\partial u}\left(u\dfrac{\partial w}{\partial u} + v\dfrac{\partial w}{\partial v}\right)e^x \sin y + \dfrac{\partial}{\partial v}\left(u\dfrac{\partial w}{\partial u} + v\dfrac{\partial w}{\partial v}\right)e^x \cos y$

$$= \left(u\frac{\partial^2 w}{\partial u^2} + \frac{\partial w}{\partial u} + v\frac{\partial^2 w}{\partial u\,\partial v}\right)u + \left(u\frac{\partial^2 w}{\partial v\,\partial u} + v\frac{\partial^2 w}{\partial v^2} + \frac{\partial w}{\partial v}\right)v$$

$\therefore \quad \dfrac{\partial^2 w}{\partial x^2} = u^2\dfrac{\partial^2 w}{\partial u^2} + u\dfrac{\partial w}{\partial u} + 2uv\dfrac{\partial^2 w}{\partial u\,\partial v} + v^2\dfrac{\partial^2 w}{\partial v^2} + v\dfrac{\partial w}{\partial v} \qquad \ldots (1)$

Next $\dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial y} = \dfrac{\partial w}{\partial u}(e^x \cos y) + \dfrac{\partial w}{\partial v}(-e^x \sin y)$

or $\dfrac{\partial w}{\partial y} = v\dfrac{\partial w}{\partial u} - u\dfrac{\partial w}{\partial v} = h\ (say)$

Again we have $\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial w}{\partial y}\right) = \dfrac{\partial h}{\partial y}\ ;\ h \to u,\ v$

$\therefore \quad \dfrac{\partial h}{\partial y} = \dfrac{\partial h}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial y} = \dfrac{\partial h}{\partial u}(e^x \cos y) + \dfrac{\partial h}{\partial v}(-e^x \sin y)$

i.e., $\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial u}\left(v\dfrac{\partial w}{\partial u} - u\dfrac{\partial w}{\partial v}\right)v + \dfrac{\partial}{\partial v}\left(v\dfrac{\partial w}{\partial u} - u\dfrac{\partial w}{\partial v}\right)(-u)$

i.e., $\dfrac{\partial^2 w}{\partial y^2} = \left(v\dfrac{\partial^2 w}{\partial u^2} - u\dfrac{\partial^2 w}{\partial u\,\partial v} - \dfrac{\partial w}{\partial v}\right)v + \left(v\dfrac{\partial^2 w}{\partial v\,\partial u} + \dfrac{\partial w}{\partial u} - u\dfrac{\partial^2 w}{\partial v^2}\right)(-u)$

i.e., $\dfrac{\partial^2 w}{\partial y^2} = v^2\dfrac{\partial^2 w}{\partial u^2} - 2uv\dfrac{\partial^2 w}{\partial u\,\partial v} - v\dfrac{\partial w}{\partial v} - u\dfrac{\partial w}{\partial u} + u^2\dfrac{\partial^2 w}{\partial v^2} \qquad \ldots (2)$

$\therefore \quad (1) + (2)$ gives, $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial^2 w}{\partial u^2}(u^2 + v^2) + \dfrac{\partial^2 w}{\partial v^2}(u^2 + v^2)$

Thus, $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (u^2 + v^2)\left(\dfrac{\partial^2 w}{\partial u^2} + \dfrac{\partial^2 w}{\partial v^2}\right)$

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59. *Transform Laplace's equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *into the polar form*

>> In polars we have $x = r\cos\theta$ and $y = r\sin\theta$

In order to find the two partial derivatives in the given equation as a composite function we need to have the composition in the form,

$$\{u \to (r,\ \theta) \to (x,\ y)\} \Rightarrow u \to x,\ y$$

$\therefore$ we have to express $r,\ \theta$ interms of x and y by simple familiar elimination.

$$x^2+y^2 = r^2 \ ; \ y/x = \tan\theta \quad \text{or} \quad \theta = \tan^{-1}(y/x)$$

We do need the partial derivatives of r and θ w.r.t x and y while using the chain rule.

Hence, $r^2 = x^2+y^2$ gives $2r\dfrac{\partial r}{\partial x} = 2x$ and $2r\dfrac{\partial r}{\partial y} = 2y$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{or} \quad \frac{\partial r}{\partial x} = \cos\theta, \ \frac{\partial r}{\partial y} = \sin\theta$$

Also $\theta = \tan^{-1}(y/x)$ gives $\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+(y/x)^2}\cdot -\dfrac{y}{x^2} = -\dfrac{y}{x^2+y^2}$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

or
$$\frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{r}, \quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

Now by chain rule, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial\theta}\dfrac{\partial\theta}{\partial x}$

i.e.,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial\theta}\left(\frac{-\sin\theta}{r}\right) = g\,(say)\ , \ g \to r,\ \theta$$

Also
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial g}{\partial x}$$

Again by chain rule we have, $\dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial g}{\partial\theta}\dfrac{\partial\theta}{\partial x}$

i.e.,
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right]\cos\theta + \frac{\partial}{\partial\theta}\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right]\left(\frac{-\sin\theta}{r}\right)$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\cos\theta - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial u}{\partial\theta}\sin\theta\right]\cos\theta$$

$$+ \left[\frac{\partial^2 u}{\partial\theta\,\partial r}\cos\theta - \frac{\partial u}{\partial r}\sin\theta - \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r} - \frac{\partial^2 u}{\partial\theta^2}\frac{\sin\theta}{r}\right]\left(\frac{-\sin\theta}{r}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \qquad (1)$$

Next $\quad \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial y}$, by chain rule.

ie., $\quad \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\sin\theta + \dfrac{\partial u}{\partial \theta}\dfrac{\cos\theta}{r} = h\,(say)\ ;\ h \to (r, \theta)$

$\therefore \quad \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial h}{\partial y}$

Again by chain rule we have $\dfrac{\partial h}{\partial y} = \dfrac{\partial h}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial h}{\partial \theta}\dfrac{\partial \theta}{\partial y}$

ie.,
$$\frac{\partial^2 u}{\partial y^2} \quad \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right]\sin\theta + \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right]\left(\frac{\cos\theta}{r}\right)$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\sin\theta + \frac{\partial^2 u}{\partial r\,\partial\theta}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r^2}\right]\sin\theta$$

$$+ \left[\frac{\partial^2 u}{\partial \theta\,\partial r}\sin\theta + \frac{\partial u}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right]\left(\frac{\cos\theta}{r}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad \ldots (2)$$

Adding (1) and (2) we have,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}(\cos^2\theta + \sin^2\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(\sin^2\theta + \cos^2\theta)$$

$$+ \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(\sin^2\theta + \cos^2\theta)$$

Thus $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ transforms into

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \text{ in the polar form.}$$

EXERCISES

1. If $u = e^{xy}$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{u}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\}$

2. If $u = e^{-x}(x\cos y + y\sin y)$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

3. If $u = \frac{1}{\sqrt{t}} e^{-x^2/4a^2 t}$ show that $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

4. If $e^{-z/x^2 - y^2} = x - y$, prove that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2 - y^2$

5. If $u = f(x + iy) + g(x - iy)$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

6. If $z = f(y + x) + g(y + 2x) + 2e^{x+2y}$ show that $z_{xx} - 3z_{xy} + 2z_{yy} = 6e^{x+2y}$

7. If $z = x f_1(x + t) + f_2(x + t)$, show that $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial t} + \frac{\partial^2 z}{\partial t^2} = 0$

8. If $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ show that $x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2y^2$

9. If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ verify that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

10. If $u = \sinh^{-1}(x/y)$ verify that $u_{xy} = u_{yx}$

11. If $x = e^u \cos v$ and $y = e^u \sin v$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

    and verify that $\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$

12. If $x = r\cos\theta$ and $y = r\sin\theta$, show that $\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x\,\partial y}\right)^2$

13. If $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

    where $u = \frac{1}{r}$

14. If $u = r^m$ where $r^2 = x^2 + y^2 + z^2$,
show that $u_{xx} + u_{yy} + u_{zz} = m(m+1)r^{m-2}$

15. If $r^2 = x^2 + y^2 + z^2$, show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{r} = 0$

16. If $u = x^2 + y^2 + z^2$ where $x = e^{2t}$, $y = e^{2t}\cos t$, $z = e^{2t}\sin t$ find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.

17. If $u = \sin(x/y)$ where $x = e^t$, $y = t^2$ find $\frac{du}{dt}$ using the rule of composite functions.

18. If $z = \cos^{-1}(x - y)$ where $x = 4t^3$, $y = 3t$ prove that $\frac{dz}{dt} = -\frac{3}{\sqrt{1-t^2}}$ and also verify the result by direct differentiation

19. If $x^3 + y^3 = 3xy$ find $\frac{dy}{dx}$ using partial derivatives

20. Find $\frac{dy}{dx}$ given the relation $x^y + y^x = a$

21. If $u = x\log(xy)$ and $x^3 + y^3 - 3xy = 0$ find $\frac{du}{dx}$

22 If $z = f(x, y)$ where $x = e^u\cos v$, $y = e^u\sin v$ prove that
$(x+y)\frac{\partial z}{\partial u} + (x-y)\frac{\partial z}{\partial v} = e^{2u}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)$

23 If $z = f(x, y)$ where $x = u^2 + v^2$, $y = 2uv$ show that
$u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = 2(u+v)(u-v)\frac{\partial z}{\partial x}$

24. If $z = f(x, y)$ where $x = u^2 - v^2$, $y = 2uv$ show that
$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 4(u^2+v^2)\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

25. If $u = f(x/z, y/z)$, prove that $xu_x + yu_y + zu_z = 0$

26. If $z = f(x, y)$ and $x = u - v$, $y = uv$, prove that
$(1+u)\frac{\partial z}{\partial u} + (1-v)\frac{\partial z}{\partial v} = 2(u+v)\frac{\partial z}{\partial y}$

27. If z is a function of x, y and u, v are connected by the relations $u = 3x + 4y$, $v = 3y - 4x$ show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 25\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

28. If $z = f(x, y)$ where $x = e^u \cos v$, $y = e^u \sin v$ prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2u}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

29. If $u = f(x, y)$ and $x = X\cos\alpha - Y\sin\alpha$, $y = X\sin\alpha + Y\cos\alpha$ prove that $u_{xx} + u_{yy} = u_{XX} + u_{YY}$

30. If $u = z^n f\left(\frac{x}{z}, \frac{y}{z}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

ANSWERS

16. $8e^{4t}$ 17. $\frac{e^t(t-2)}{t^3}\cos\left(\frac{e^t}{t^2}\right)$ 19. $\frac{y-x^2}{y^2-x}$

20. $\frac{-y(x^{y-1}+y^{x-1}\log y)}{x(y^{x-1}+x^{y-1}\log x)}$ 21. $\log(exy) + \frac{x(x^2-y)}{y(x-y^2)}$

3.2 Jacobians

3.21 Introduction

This topic is basically involved with the evaluation of determinants of second order and third order whose elements are represented by first order partial derivatives of two or three given functions / transformations.

The application of *Jacobians* is significant in the evaluation of double integrals of the form $\iint f(x, y)\,dx\,dy$ and triple integrals of the form $\iiint f(x, y, z)\,dx\,dy\,dz$ by transformation from one system (*set*) of coordinates (*variables*) to the other. The principle of evaluation is analogous with the evaluation of $\int f(x)\,dx$ by taking a suitable substitution.

3.22 Definition

Let u and v be functions of two independent variables x and y. The *Jacobian* (J) of u and v *w.r.t.* x and y is symbolically represented and defined as follows.

$$J\left(\frac{u,\ v}{x,\ y}\right) \text{ or } \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Similarly if $u,\ v,\ w$ are functions of three independent variables $x,\ y,\ z$ then

$$J\left(\frac{u,\ v,\ w}{x,\ y,\ z}\right) \text{ or } \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

WORKED PROBLEMS

60. *Find the Jacobian of $u,\ v,\ w$ w.r.t. $x,\ y,\ z$ given that $u = x+y+z,\ v = y+z,\ w = z$*

$>>$ We have to find $J = \dfrac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$

But $\quad u = x+y+z,\ v = y+z,\ w = z$

$$\therefore \quad J = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding by the last row we get

$J = 1(1-0) = 1 \qquad$ Thus $J = 1$

---

61. *Find* $\dfrac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ *where* $u = x^2+y^2+z^2,\ v = xy+yz+zx,\ w = x+y+z$

$>>$ The definition of $J = \dfrac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ is same as in the previous problem.

But $\quad u = x^2+y^2+z^2,\ v = xy+yz+zx,\quad w = x+y+z$

Substituting for the partial derivatives we get

$$J = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}.$$

Expanding by the first row,

$$J = 2x\,\{(x+z)-(y+x)\} - 2y\,\{(y+z)-(y+x)\} + 2z\,\{(y+z)-(x+z)\}$$

$$= 2x(z-y) - 2y(z-x) + 2z(y-x)$$

$$= 2(xz - xy - yz + xy + yz - xz) = 0 \qquad \text{Thus } J = 0$$

Aliter : *(By using properties of determinants)*

$$J = 2\begin{vmatrix} x & y & z \\ (y+z) & (x+z) & (y+x) \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & y & z \\ (x+y+z) & (x+y+z) & (x+y+z) \\ 1 & 1 & 1 \end{vmatrix} \cdots R_2 \to R_1 + R_2$$

$$J = 2(x+y+z)\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \quad \because \text{ two rows are identical.}$$

---

2. [illegible]

$$>> \quad J\left(\frac{u,\ v,\ w}{x_1,\ x_2,\ x_3}\right) = \begin{vmatrix} \dfrac{\partial u}{\partial x_1} & \dfrac{\partial u}{\partial x_2} & \dfrac{\partial u}{\partial x_3} \\ \dfrac{\partial v}{\partial x_1} & \dfrac{\partial v}{\partial x_2} & \dfrac{\partial v}{\partial x_3} \\ \dfrac{\partial w}{\partial x_1} & \dfrac{\partial w}{\partial x_2} & \dfrac{\partial w}{\partial x_3} \end{vmatrix}$$

where $u = \sqrt{x_1 x_2}$ $v = \sqrt{x_2 x_3}$, $w = \sqrt{x_3 x_1}$

$$= \begin{vmatrix} \dfrac{x_2}{2\sqrt{x_1 x_2}} & \dfrac{x_1}{2\sqrt{x_1 x_2}} & 0 \\ 0 & \dfrac{x_3}{2\sqrt{x_2 x_3}} & \dfrac{x_2}{2\sqrt{x_2 x_3}} \\ \dfrac{x_3}{2\sqrt{x_3 x_1}} & 0 & \dfrac{x_1}{2\sqrt{x_3 x_1}} \end{vmatrix}$$

$$= \frac{x_2}{2\sqrt{x_1 x_2}}\left\{\frac{x_1 x_3}{4x_3\sqrt{x_1 x_2}} - 0\right\} - \frac{x_1}{2\sqrt{x_1 x_2}}\left\{0 - \frac{x_2 x_3}{4x_3\sqrt{x_1 x_2}}\right\}$$

$$= \frac{x_1 x_2 x_3}{8x_1 x_2 x_3} + \frac{x_1 x_2 x_3}{8x_1 x_2 x_3} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \qquad \text{Thus } J = \mathbf{\frac{1}{4}}$$

Aliter : $J = \dfrac{1}{2\sqrt{x_1 x_2}} \cdot \dfrac{1}{2\sqrt{x_2 x_3}} \cdot \dfrac{1}{2\sqrt{x_3 x_1}} \begin{vmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \end{vmatrix}$

$$= \frac{1}{8(x_1 x_2 x_3)}\{x_2(x_1 x_3 - 0) - x_1(0 - x_2 x_3)\} = \frac{2x_1 x_2 x_3}{8x_1 x_2 x_3} = \mathbf{\frac{1}{4}}$$

---

63. *If $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$, evaluate the jacobian of x, y, z w.r.t. ρ, ϕ, z*

>> $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding by the last row, $J = 1(\rho\cos^2\phi + \rho\sin^2\phi) = \rho$

---

64. *If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ show that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$*

>> $$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}$$

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi\ \ \ z = r\cos\theta$$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

On expanding by the last row we get,

$$\cos\theta \cdot r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi$$
$$+ r\sin\theta\left\{r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right\}$$
$$= r^2\sin\theta\cos^2\theta(\cos^2\phi+\sin^2\phi) + r^2\sin^3\theta(\cos^2\phi+\sin^2\phi)$$
$$= r^2\sin\theta\cos^2\theta \cdot 1 + r^2\sin^3\theta \cdot 1 = r^2\sin\theta(\cos^2\theta+\sin^2\theta)$$

Thus $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \boldsymbol{r^2\sin\theta}$

Aliter : Taking common factors r, $r\sin\theta$ respectively in the second and third columns we have

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r \cdot r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Expanding by the last row we have

$$r^2\sin\theta\left\{\cos\theta(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)\right.$$
$$\left.+ \sin\theta(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)\right\}$$
$$= r^2\sin\theta\left[\cos^2\theta(\cos^2\phi+\sin^2\phi) + \sin^2\theta(\cos^2\phi+\sin^2\phi)\right]$$
$$= r^2\sin\theta\left[\cos^2\theta \cdot 1 + \sin^2\theta \cdot 1\right] = r^2\sin\theta \cdot 1 = \boldsymbol{r^2\sin\theta}$$

---

65. *If* $x+y+z = u$, $y+z = v$ *and* $z = uvw$, *find the value of* $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$

>> $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

It is evident that we should have x, y, z in terms of u, v, w

Consider $x+y+z = u$ ... (1), $y+z = v$ ... (2), $z = uvw$ ... (3)

Using (2) in (1) we have, $x+v = u$ $\therefore$ $x = u-v$

Also by using (3) in (2) we have, $y + uvw = v \quad \therefore \quad y = v - uvw$

Thus the given data is modified into the form,

$$x = u - v, \;\; y = v - uvw, \;\; z = uvw$$

$$\therefore \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & (1 - uw) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1\,[\,(1 - uw)\,uv - (uw)(-uv)\,] + 1\,[\,(-vw)(uv) - (vw)(-uv)\,]$$

$$= uv - u^2 vw + u^2 vw - uv^2 w + uv^2 w = uv$$

Thus $J = uv$

Aliter : Adding R_3 to R_2 we get,

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ vw & uw & uv \end{vmatrix}$$

On expanding by the second row we have

$$-0 + 1\,(uv - 0) - 0 = uv$$

---

66. *If* $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$, $w = \dfrac{xy}{z}$, *show that* $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

>> By data $u = \dfrac{yz}{x},\; v = \dfrac{zx}{y},\; w = \dfrac{xy}{z}$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2mm] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \dfrac{-yz}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\[2mm] \dfrac{z}{y} & \dfrac{-zx}{y^2} & \dfrac{x}{y} \\[2mm] \dfrac{y}{z} & \dfrac{x}{z} & \dfrac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left\{\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{x}{z}\right)\left(\frac{x}{y}\right)\right\}$$

$$-\frac{z}{x}\left\{\frac{z}{y}\left(\frac{-xy}{z^2}\right) - \frac{y}{z}\cdot\frac{x}{y}\right\} + \frac{y}{x}\left\{\frac{z}{y}\cdot\frac{x}{z} - \frac{y}{z}\left(\frac{-zx}{y^2}\right)\right\}$$

$$= \frac{-yz}{x^2}\left\{\frac{x^2}{yz} - \frac{x^2}{yz}\right\} - \frac{z}{x}\left\{\frac{-x}{z} - \frac{x}{z}\right\} + \frac{y}{x}\left\{\frac{x}{y} + \frac{x}{y}\right\}$$

$$= 0 + 1 + 1 + 1 + 1 = 4$$

Thus $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

Aliter : We shall avoid denominators in every element by removal of $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$ respectively from the first, second and the third row

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2 y^2 z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} \cdots R_2 \to R_1 + R_2 \\ \cdot R_3 \to R_1 + R_3 \end{matrix}$$

$$= 1 \cdot \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -1(0-4) = 4$$

---

67. *If* $u = x + 3y^2 - z^3$, $v = 4x^2 yz$, $w = 2z^2 - xy$ *find* $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ *at* $(1, -1, 0)$

\>> $\quad u = x + 3y^2 - z^3 \; , \; v = 4x^2 yz \; , \; w = 2z^2 - xy$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2 z & 4x^2 y \\ -y & -x & 4z \end{vmatrix}$$

It will be easier if the elements of the determinant are evaluated at $(1, -1, 0)$

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} \text{ at } (1, -1, 0) = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

Expanding by the second row we get, $-0 + 0 + 4(-1 + 6) = 20$

---

68. *If* $u = x(1 - r^2)^{-1/2}$, $v = y(1 - r^2)^{-1/2}$ *and* $r^2 = x^2 + y^2$ *find* $J\left(\begin{matrix} u, v \\ x, y \end{matrix}\right)$

\>> $$J\left(\begin{matrix} u, v \\ x, y \end{matrix}\right) \text{ or } \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Consider $u = x(1-r^2)^{-1/2}$

$$\therefore \quad \frac{\partial u}{\partial x} = x \cdot \frac{-1}{2}(1-r^2)^{-3/2} \cdot -2r\frac{\partial r}{\partial x} + 1 \cdot (1-r^2)^{-1/2}$$

$$= x(1-r^2)^{-3/2}\, r\, \frac{\partial r}{\partial x} + (1-r^2)^{-1/2}$$

But $\quad r^2 = x^2 + y^2.$

Differentiating partially *w.r.t* x, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{Similarly} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Now $\quad \dfrac{\partial u}{\partial x} = x(1-r^2)^{-3/2}\, r \cdot \dfrac{x}{r} + (1-r^2)^{-1/2}$

i.e., $\quad \dfrac{\partial u}{\partial x} = (1-r^2)^{-3/2}\, x^2 + (1-r^2) \quad$ But $x^2 = r^2 - y^2$

$$\frac{\partial u}{\partial x} = (1-r^2)^{-3/2}(1-y^2) \qquad \ldots (1)$$

Also $\quad \dfrac{\partial u}{\partial y} = x \cdot -\dfrac{1}{2}(1-r^2)^{-3/2} \cdot -2r\dfrac{\partial r}{\partial y}$

$$\therefore \quad \frac{\partial u}{\partial y} = x(1-r^2)^{-3/2}\, r \cdot \frac{y}{r} = xy(1-r^2)^{-3/2} \qquad \ldots (2)$$

Similarly we can obtain for $v = y(1-r^2)^{-1/2}$

$$\frac{\partial v}{\partial x} = xy(1-r^2)^{-3/2} \qquad \ldots (3)$$

$$\frac{\partial v}{\partial y} = (1-r^2)^{-3/2}(1-x^2) \qquad \ldots (4)$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} (1-r^2)^{-3/2}(1-y^2) & xy(1-r^2)^{-3/2} \\ xy(1-r^2)^{-3/2} & (1-r^2)^{-3/2}(1-x^2) \end{vmatrix}$$

Now taking $(1-r^2)^{-3/2}$ as a common factor in each row of the determinant we have,

$$\frac{\partial(u, v)}{\partial(x, y)} = (1-r^2)^{-3/2} \cdot (1-r^2)^{-3/2} \begin{vmatrix} (1-y^2) & xy \\ xy & (1-x^2) \end{vmatrix}$$

$$= (1-r^2)^{-3}\left\{(1-y^2)(1-x^2) - x^2y^2\right\}$$

$$= (1-r^2)^{-3}\left\{1 - x^2 - y^2 + x^2y^2 - x^2y^2\right\}$$

$$= (1-r^2)^{-3}\left\{1-(x^2+y^2)\right\} = (1-r^2)^{-3}(1-r^2) = (1-r^2)^{-2}$$

Thus $\quad \dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{(1-r^2)^2}$

69. *If* $x = a\cosh u \cos v$ *and* $y = a \sinh u \sin$

prove that $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{a^2}{2}(\cosh 2u - \cos 2v)$

\>> We have $x = a\cosh u\cos v,\ y = a\sinh u \sin v$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a\sinh u\cos v & -a\cosh u \sin v \\ a\cosh u\sin v & a\sinh u\cos v \end{vmatrix}$$

ie.,

$$= a^2(\sinh^2 u\cos^2 v + \cosh^2 u \sin^2 v)$$

$$= \frac{a^2}{2}(\sinh^2 u \cdot 2\cos^2 v + \cosh^2 u \cdot 2\sin^2 v)$$

But $\quad 1+\cos 2v = 2\cos^2 v$ and $1-\cos 2v = 2\sin^2 v$

$$\therefore \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{a^2}{2}\left\{\sinh^2 u(1+\cos 2v) + \cosh^2 u(1-\cos 2v)\right\}$$

$$= \frac{a^2}{2}\left\{(\cosh^2 u + \sinh^2 u) - \cos 2v(\cosh^2 u - \sinh^2 u)\right\}$$

But $\cosh^2 u - \sinh^2 u = 1$ and $\cosh^2 u + \sinh^2 u = \cosh 2u$.

Thus, $\quad \dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{a^2}{2}(\cosh 2u - \cos 2v)$

70. *If* $u+v = e^x\cos y$ *and* $u-v = e^x \sin y$ *find the jacobian of the functions* u *and* v *w.r.t.* x *and* y

\>> We have to find $\dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

Using the given data we have to solve for u and v interms of x and y

By data $\quad u+v = e^x\cos y \qquad \ldots(1)$

$\quad u-v = e^x \sin y \qquad \ldots(2)$

$$(1)+(2) \text{ gives}: \; 2u = e^x(\cos y + \sin y)$$

$$(1)-(2) \text{ gives}: \; 2v = e^x(\cos y - \sin y)$$

i.e., $u = \dfrac{e^x}{2}(\cos y + \sin y)$, $v = \dfrac{e^x}{2}(\cos y - \sin y)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{e^x}{2}(\cos y + \sin y), \quad \frac{\partial v}{\partial x} = \frac{e^x}{2}(\cos y - \sin y)$$

$$\frac{\partial u}{\partial y} = \frac{e^x}{2}(-\sin y + \cos y), \quad \frac{\partial v}{\partial y} = \frac{e^x}{2}(-\sin y - \cos y)$$

Now
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{e^x}{2}(\cos y + \sin y) & \dfrac{e^x}{2}(-\sin y + \cos y) \\ \dfrac{e^x}{2}(\cos y - \sin y) & -\dfrac{e^x}{2}(\sin y + \cos y) \end{vmatrix}$$

$$= \frac{e^x}{2} \cdot \frac{e^x}{2}\left[-(\cos y + \sin y)^2 - (\cos y - \sin y)^2\right],$$

$$= \frac{-e^{2x}}{4}\left[(1 + \sin 2y) + (1 - \sin 2y)\right] = \frac{-e^{2x}}{2}$$

Thus $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{-e^{2x}}{2}$

EXERCISES

Find the Jacobian of the following functions 1 to 5

1. $u = xy^2, \; v = yz^2, \; w = zx^2$
2. $u = x\cos y\cos z, \; v = x\cos y\sin z, \; w = x\sin y$
3. $u = \dfrac{y^2}{2x}, \; v = \dfrac{x^2+y^2}{2x}$
4. $x = r\cos\theta\cos\phi, \; y = r\cos\theta\sin\phi, \; z = r\sin\theta$
5. $u = \dfrac{x}{y-z}, \; v = \dfrac{y}{z-x}, \; w = \dfrac{z}{x-y}$
6. If $u = xyz, \; v = xy + yz + zx, \; w = x + y + z$

   show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = (x-y)(y-z)(z-x)$

7. If $u = \dfrac{x}{\sqrt{1-r^2}}$, $v = \dfrac{y}{\sqrt{1-r^2}}$, $w = \dfrac{z}{\sqrt{1-r^2}}$ where $r = \sqrt{x^2+y^2+z^2}$, prove that $J\left(\dfrac{u,\, v,\, w}{x,\, y,\, z}\right) = \dfrac{1}{(1-r^2)^{5/2}}$

8. Show that for the functions $x = u^2 - v^2,\ y = v^2 - w^2,\ z = w^2 - u^2$
$J\left(\dfrac{x,\, y,\, z}{u,\, v,\, w}\right) = 0$

9. Given that $ux_1 = 2x_2 x_3,\ vx_2 = 2x_3 x_1,\ wx_3 = 2x_1 x_2$ show that
$\dfrac{\partial(u,\, v,\, w)}{\partial(x_1,\, x_2,\, x_3)} = 96$

10. If $x + y + z = u,\ y + z = uv,\ z = uvw$ then show that $\dfrac{\partial(x,\, y,\, z)}{\partial(u,\, v,\, w)} = u^2 v$

ANSWERS

1. $9x^2 y^2 z^2$ 2. $-x^2 \cos y$ 3. $-y/2x$

4. $r^2 \cos\theta$ 5. 0

3.3 Taylor's theorem for a function of two variables

Statement [illegible] (a, b) then

$$f(a+h,\, b+k) = f(a,\, b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,\, b)$$
$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,\, b) + \ldots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,\, b)$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h,\, b+\theta k) \quad \text{where } 0 < \theta < 1$$

3.31 Taylor's series for $f(x,\, y)$

We denote $R_n = \dfrac{1}{n!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^n f(a+\theta h,\, b+\theta k)$ being the remainder after n terms.

Let $a + h = x$, $b + k = y$ ie., $h = x - a$, $k = y - b$

If x is close enough to a and y to b, h, k will be very small and $R_n \to 0$ as $n \to \infty$

As $n \to \infty$ the number of terms increase indefinitely and we have an infinite series expansion of $f(x, y)$ in powers of $(x-a)$, $(y-b)$ referred to as the **Taylor's series of $f(x, y)$ about (a, b)** which is as follows

$$f(x, y) = f(a, b) + \frac{1}{1!}\left\{(x-a)f_x(a, b) + (y-b)f_y(a, b)\right\}$$

$$+ \frac{1}{2!}\left\{(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\right\} + \cdots \qquad \ldots (1)$$

In particular if $(a, b) = (0, 0)$, the series is called as **Taylor's series about the origin or Maclaurin's series for $f(x, y)$** given by

$$f(x, y) = f(0, 0) + \frac{1}{1!}\left\{x f_x(0, 0) + y f_y(0, 0)\right\}$$

$$+ \frac{1}{2!}\left\{x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right\} + \ldots \qquad \ldots (2)$$

The remainder after n terms in this case will be

$$R_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y) \quad \text{and} \quad R_n \to 0 \quad \text{as} \quad n \to \infty$$

Further if the Taylor's series of $f(x, y)$ is approximated to some terms upto a particular degree the resulting expression of $f(x, y)$ is called as the Taylor's polynomial.

ILLUSTRATIVE EXAMPLES

1. Taylor's expansion of $e^x \sin y$ about $(1, \pi/4)$

>> The expansion of $f(x, y)$ about $(1, \pi/4)$ is given by

$$f(x, y) = f(1, \pi/4) + (x-1)f_x(1, \pi/4) + (y-\pi/4)f_y(1, \pi/4)$$

$$+ \frac{1}{2!}\left\{(x-1)^2 f_{xx}(1, \pi/4) + 2(x-1)(y-\pi/4)f_{xy}(1, \pi/4) + (y-\pi/4)^2 f_{yy}(1, \pi/4)\right\} + \cdots$$

The given function and its partial derivatives evaluated at $(1, \pi/4)$ is as follows

$$f(x, y) = e^x \sin y \to e/\sqrt{2}$$

$$f_x = e^x \sin y \to e/\sqrt{2}$$

$$f_y = e^x \cos y \to e/\sqrt{2}$$

$$f_{xx} = e^x \sin y \to e/\sqrt{2}$$

$$f_{xy} = e^x \cos y \to e/\sqrt{2}$$

$$f_{yy} = -e^x \sin y \to -e/\sqrt{2}$$

Substituting these values in the expansion of $f(x, y)$ we have,

$$e^x \sin y = \frac{e}{\sqrt{2}}\left[1+\left\{(x-1)+(y-\pi/4)\right\}\right.$$
$$\left.+\frac{1}{2!}\left\{(x-1)^2+2(x-1)(y-\pi/4)-(y-\pi/4)^2\right\}+\cdots\right]$$

2. Expansion of xy^2+x^2y in powers of $(x-1)$ and $(y+3)$

>> The point (a, b) so as to obtain the expansion in powers of $(x-1)$ and $y+3$ is obviously $(1, -3)$ The associated expansion of $f(x, y)$ about $(1, -3)$ is given by

$$f(x, y) = f(1, -3)+\left\{(x-1)f_x(1, -3)+(y+3)f_y(1, -3)\right\}$$
$$+\frac{1}{2!}\left\{(x-1)^2 f_{xx}(1, -3)+2(x-1)(y+3)f_{xy}(1, -3)\right.$$
$$\left.+(y+3)^2 f_{yy}(1, -3)\right\}+\cdots$$

The given function and its partial derivatives evaluated at $(1, -3)$ is as follows.

$$f(x, y) = xy^2+x^2y \to 6$$

$$f_x = y^2+2xy \to 3$$

$$f_y = 2xy+x^2 \to -5$$

$$f_{xx} = 2y \to -6$$

$$f_{xy} = 2y+2x \to -4$$

$$f_{yy} = 2x \to 2$$

Substituting these values in the expansion of $f(x, y)$ we have,

$$xy^2+x^2y = 6+\left\{3(x-1)-5(y+3)\right\}+\frac{1}{2!}\left\{-6(x-1)^2\right.$$
$$\left.-8(x-1)(y+3)+2(y+3)^2\right\}+\cdots$$

3. Taylor's expansion of $e^x \log(1+y)$ about the origin.

Taylor's expansion of $f(x, y)$ about $(0, 0)$ is given by

$$f(x, y) = f(0, 0) + \{x f_x(0, 0) + y f_y(0, 0)\}$$

$$+ \frac{1}{2!}\left\{x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right\} + \cdots$$

The given function and its partial derivatives evaluated at $(0, 0)$ is as follows.

$$\begin{aligned}
f(x, y) &= e^x \log(1+y) && \to 0 \\
f_x &= e^x \log(1+y) && \to 0 \\
f_y &= e^x \cdot \frac{1}{1+y} && \to 1 \\
f_{xx} &= e^x \log(1+y) && \to 0 \\
f_{xy} &= e^x \cdot \frac{1}{1+y} && \to 1 \\
f_{yy} &= e^x \cdot \frac{-1}{(1+y)^2} && \to -1
\end{aligned}$$

Substituting these values in the expansion of $f(x, y)$ we have,

$$e^x \log(1+y) = y + \frac{1}{2!}(2xy - y^2) + \cdots$$

Maxima & Minima for a function of two variables

First, we briefly recapitulate the concept of maxima and minima for a function of one variable.

A function $f(x)$ is said to have a maximum value at a point $x = a$ if there exists a neighbourhood of the point 'a' say $(a+h)$, h is small, such that $f(a) > f(a+h)$. Similarly if $f(a) < f(a+h)$ then $f(x)$ is said to have a minimum value at $x = a$. $f(a)$ is said to be an extreme value of $f(x)$ if it is either a maximum or a minimum.

A necessary condition for $f(a)$ to be an extreme value of $f(x)$ is that $f'(a) = 0$. Also $f(a)$ is a maximum value of $f(x)$ if $f'(a) = 0$ and $f''(a) < 0$ & $f(a)$ is a minimum value of $f(x)$ if $f'(a) = 0$ and $f''(a) > 0$.

We discuss these concepts concerning a function of two variables.

Definition A function $f(x, y)$ is said to have a **maximum value** at a point (a, b) if there exists a neighbourhood of the point (a, b) say $(a+h, b+k)$, h and k are small such that $f(a, b) > f(a+h, b+k)$

Similarly if $f(a, b) < f(a+h, b+k)$ then $f(x, y)$ is said to have a **minimum value** at (a, b). Also $f(a, b)$ is said to be an **extreme value** of $f(x, y)$ if it is a maximum value or a minimum value.

·41 Necessary and suffecient conditions for maxima or minima

Necessary conditions : If $f(a, b)$ is an extreme value of $f(x, y)$ then it has to be an extreme value of the function of a single variable $f(x, b)$ and also of a function of a single variable $f(a, y)$. Hence the first derivatives of these two functions of a single variable must be zero as per the necessary condition of a function of a single variable.

ie., $\dfrac{d}{dx}[f(x, b)] = 0$ at $x = a$ and $\dfrac{d}{dy}[f(a, y)] = 0$ at $y = b$

In other words, these are the partial derivatives of $f(x, y)$ w.r.t x and w.r.t y at (a, b)

ie., $\quad f_x(a, b) = 0$ **and** $f_y(a, b) = 0.$

These are the necessary conditions.

Suffecient conditions : Let $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Let us consider Taylor's theorem for $f(x, y)$ in the form,

$$f(a+h, b+k) = f(a, b) + \left\{ h f_x(a,b) + k f_y(a, b) \right\}$$
$$+ \frac{1}{2!}\left\{ h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b) \right\}$$

by neglecting other higher order terms for suffeciently small h and k.

Using $f_x(a, b) = 0$, $f_y(a, b) = 0$ and by denoting

$A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$ we have,

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}(A h^2 + 2 B h k + C k^2)$$
$$= \frac{1}{2A}(A^2 h^2 + 2 A B h k + AC k^2)$$
$$= \frac{1}{2A}[(Ah + Bk)^2 + (AC - B^2) k^2]$$

ie., $$f(a+h, b+k) - f(a, b) = \frac{\Delta}{2A} \ (say) \qquad \ldots(1)$$

where $\Delta = (A h + B k)^2 + (AC - B^2) k^2$

Clearly $\Delta > 0$ if $AC - B^2 > 0$ and $\Delta < 0$ if $AC - B^2 < 0$.

Further $f(a+h,\ b+k) - f(a,\ b) > 0$ if $AC - B^2 > 0,\ A > 0$

and $f(a+h,\ b+k) - f(a,\ b) < 0$ if $AC - B^2 > 0,\ A < 0$

$\Rightarrow$ $f(a+h,\ b+k) > f(a,\ b)$ if $AC - B^2 > 0$ and $A > 0$ and
$f(a+h,\ b+k) < f(a,\ b)$ if $AC - B^2 > 0$ and $A < 0$

or $f(a,\ b) > f(a+h,\ b+k)$ if $AC - B^2 > 0,\ A < 0$ and
$f(a,\ b) < f(a+h,\ b+k)$ if $AC - B^2 > 0,\ A > 0$

Taking a note of the definition for maxima-minima we conclude that ,
$f(a,\ b)$ is a maximum value if $AC - B^2 > 0$ and $A < 0$

Conclusion : *Thus the necessary and the suffecient conditions for* $f(x,\ y)$ *to have a maximum value at* $(a,\ b)$ *is* $f_x(a,\ b) = 0$, $f_y(a,\ b) = 0$ *and* $AC - B^2 > 0$, $A < 0$. *Also for a minimum value,* $f_x(a,\ b) = 0$, $f_y(a,\ b) = 0$ *and* $AC - B^2 > 0,\ A > 0$

Note : *If* $AC - B^2 < 0$, $f(a,\ b)$ *is not an extreme value.* $f(x,\ y)$ *is neither a maximum nor a minimum at the point* $(a,\ b)$ *and the point is called a saddle point.*

If $AC - B^2 = 0$ *then we have,* $f(a+h,\ b+k) - f(a,\ b) = (Ah + Bk)^2/2A$

R.H.S is positive or negative according as $A > 0$ *or* $A < 0$. *However if* $Ah + Bk = 0$, *then* $h/k = -B/A$ *and the case is doubtful. Further discussion is needed based on various geometrical factors*

Working procedure for finding extreme values of $f(x,\ y)$

- We have to first find the stationary points $(x,\ y)$ such that $f_x = 0$ and $f_y = 0$
- We then find the second order partial derivatives
  $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$. We evaluate these at all the stationary points and also compute the corresponding value of $AC - B^2$
- (a) A stationary point $(x_0,\ y_0)$ is a maximum point if
  $AC - B^2 > 0$ & $A < 0$. $f(x_0,\ y_0)$ is a maximum value
  (b) A stationary point $(x_1,\ y_1)$ is a minimum point if
  $AC - B^2 > 0$ & $A > 0$. $f(x_1,\ y_1)$ is a minimum value.

Note : *We can overlook the cases of* $AC - B^2 < 0$, $AC - B^2 = 0$, $A = 0$

WORKED PROBLEMS

71. *Find the extreme values of the function* $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

>> $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$, $f_y = 3y^2 - 12$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 - 3 = 0$ **and** $3y^2 - 12 = 0$ **or** $x^2 - 1 = 0$ **and** $y^2 - 4 = 0$

ie., $x = \pm 1$, $y = \pm 2$ $\therefore$ $(1, 2), (1, -2), (-1, 2), (-1, -2)$ are the stationary points. Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	$(1, 2)$	$(1, -2)$	$(-1, 2)$	$(-1, -2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. pt.	Saddle pt.	Saddle pt.	Max. pt.

Maximum value of $f(x, y)$ **is**

$f(-1, -2) = -1 - 8 + 3 + 24 + 20 = \mathbf{38}$

Minimum value of $f(x, y)$ **is** $f(1, 2) = 1 + 8 - 3 - 24 + 20 = \mathbf{2}$

72. *Find the extreme values of the function* $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

>> $f_x = 3x^2 + 3y^2 - 6x$; $f_y = 6xy - 6y$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 + 3y^2 - 6x = 0$ or $x^2 + y^2 - 2x = 0$ ... (1)

$6xy - 6y = 0$ or $y(x - 1) = 0$ ... (2)

$\Rightarrow$ $y = 0$, $x = 1$ from (2).

From (1) if $y = 0$ then $x = 0, 2$ and if $x = 1$, $y = \pm 1$

$\therefore$ the stationary points are $(0, 0), (2, 0), (1, 1), (1, -1)$

Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	(0, 0)	(2, 0)	(1, 1)	(1, −1)
$A = 6x - 6$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	-6
$C = 6x - 6$	-6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max pt.	Min pt.	Saddle pt.	Saddle pt

Maximum value of $f(x, y)$ is $f(0, 0) = 4$ and minimum value of $f(x, y)$ is $f(2, 0) = 0$

73. *Find the maximum and minimum values of the function $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$*

>> Let $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$f_x = 3x^2 + 3y^2 - 30x + 72 \ , \ f_y = 6xy - 30y$$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 + 3y^2 - 30x + 72 = 0$ **or** $x^2 + y^2 - 10x + 24 = 0$ ...(1)

$6xy - 30y = 0$ or $y(x-5) = 0$ ...(2)

(2) gives us $y = 0$ and $x = 5$

Putting $y = 0$ in (1) we get $x^2 - 10x + 24 = 0$

or $(x-4)(x-6) = 0$ *ie.,* $x = 4, 6$

$\therefore$ $(4, 0), (6, 0)$ are stationary points.

Putting $x = 5$ in (1) we get $y^2 - 1 = 0$ or $y = \pm 1$

$\therefore$ $(5, 1), (5, -1)$ are also stationary points.

Let us examine these points for maxima and minima

Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	(4, 0)	(6, 0)	(5, 1)	(5, −1)
$A = 6x - 30$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	-6
$C = 6x - 30$	-6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max. pt	Min. pt.	Saddle pt.	Saddle pt

Maximum value of $f(x, y)$ is $f(4, 0) = 64 - 240 + 288 = 112$
Minimum value of $f(x, y)$ is $f(6, 0) = 216 - 540 + 432 = 108$

74. *Examine the function* $xy(a - x - y)$ *for extreme values*

>> Let $f(x, y) = axy - x^2y - xy^2$

$$f_x = ay - 2xy - y^2, \ f_y = ax - x^2 - 2xy$$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

i.e., $y(a - 2x - y) = 0$ and $x(a - x - 2y) = 0$

Points (x, y) are obtained from the following set of equations

(i) $y = 0, \ x = 0 \Rightarrow (x, y) = (0, 0)$

(ii) $y = 0, \ a - x - 2y = 0 \Rightarrow (x, y) = (a, 0)$

(iii) $a - 2x - y = 0, \ x = 0 \Rightarrow (x, y) = (0, a)$

(iv) $a - 2x - y = 0, \ a - x - 2y = 0 \Rightarrow (x, y) = (a/3, a/3)$ on solving

We shall examine these points for maxima and minima.

Let $A = f_{xx}, \ B = f_{xy}, \ C = f_{yy}$

	$(0, 0)$	$(a, 0)$	$(0, a)$	$(a/3, a/3)$
$A = -2y$	0	0	$-2a$	$-2a/3$
$B = a - 2x - 2y$	a	$-a$	$-a$	$-a/3$
$C = -2x$	0	$-2a$	0	$-2a/3$
$AC - B^2$	$-a^2 < 0$	$-a^2 < 0$	$-a^2 < 0$	$a^2/3 > 0$
Conclusion	Saddle pt	Saddle pt	Saddle pt	Dependents on a

We note that $AC - B^2 > 0$ and $A = -2a/3$ is positive or negative according as a is negative or positive.

Hence $(a/3, a/3)$ is a minimum point if $a < 0$ and the same is a maximum point if $a > 0$

Now $f(a/3, a/3) = a^3/27$ **will be the extreme value of** $f(x, y)$ being minimum when $a < 0$ and maximum when $a > 0$.

75. Find the maximum and minimum values of the function $x^3 + y^3 - 63(x + y) + 12xy$

>> Let $f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$

$$f_x = 3x^2 - 63 + 12y, f_y = 3y^2 - 63 + 12x$$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

ie., $3x^2 - 63 + 12y = 0$ or $x^2 - 21 + 4y = 0$

$3y^2 - 63 + 12x = 0$ or $y^2 - 21 + 4x = 0$

On subtracting we have,

$$(x^2 - y^2) + 4(y - x) = 0 \text{ or } (x - y)(x + y - 4) = 0$$

$\therefore \quad x = y$ or $x + y = 4$

Putting $x = y$ in $x^2 - 21 + 4y = 0$ we have,

$$y^2 + 4y - 21 = 0 \text{ or } (y+7)(y-3) = 0 \text{ or } y = -7, 3$$

Since $x = y$, $(-7, -7)$ and $(3, 3)$ are stationary points.

Also $x + y = 4$ gives $x = 4 - y$ and $x^2 - 21 + 4y = 0$ becomes,

$(4 - y)^2 - 21 + 4y = 0$ or $y^2 - 4y - 5 = 0$ or $(y - 5)(y + 1) = 0$ or $y = 5, -1$

Since $x = 4 - y$, $(-1, 5)$ and $(5, -1)$ are also stationary points

Let us examine these four points for maxima and minima

Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	$(-7, -7)$	$(3, 3)$	$(-1, 5)$	$(5, -1)$
$A = 6x$	$-42 < 0$	$18 > 0$	-6	30
$B = 12$	12	12	12	12
$C = 6y$	-42	18	30	-6
$AC - B^2$	$1620 > 0$	$180 > 0$	$-324 < 0$	$-324 < 0$
Conclusion	Max pt	Min pt	Saddle pt	Saddle pt

Maximum value of $f(x, y) = f(-7, -7) = 784$

Minimum value of $f(x, y) = f(3, 3) = -216$

76. *Examine the function $f(x, y) = x^4 + y^4 - 2(x - y)^2$ for extreme values*

>> $f_x = 4x^3 - 4(x - y)$; $f_y = 4y^3 + 4(x - y)$

We have to solve $f_x = 0$, $f_y = 0$ simultaneously.

$$x^3 - (x - y) = 0 \qquad \ldots (1)$$

$$y^3 + (x - y) = 0 \qquad \ldots (2)$$

From (1) $x - y = x^3$ and we shall use this in (2)

$\therefore \quad x^3 + y^3 = 0$ or $(x+y)(x^2 - xy + y^2) = 0$

$\Rightarrow \quad y = -x \; ; \; x^2 - xy + y^2 = 0$

Using $y = -x$ in (1) we get, $x^3 - 2x = 0$ or $x(x^2 - 2) = 0$

$\therefore \quad x = 0, \; x = \pm\sqrt{2}. \quad y = -x, \; \Rightarrow y = 0, -\sqrt{2}, \sqrt{2}.$

Hence a set of stationary points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

Also $x^2 - xy + y^2 = 0$ *ie.,* $x(x - y) + y^2 = 0$ and from (1) $x^3 = (x - y)$

$\therefore$ we have $x^4 + y^2 = 0$ which clearly shows that we do not have any real values satisfying the identity.

We shall now examine the stationary points for maxima - minima.

Let $\quad A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	$(0, 0)$	$(\sqrt{2}, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$
$A = 12x^2 - 4$	-4	$20 > 0$	$20 > 0$
$B = 4$	4	4	4
$C = 12y^2 - 4$	-4	20	20
$AC - B^2$	0	$384 > 0$	$384 > 0$
Conclusion	-	Min. pt.	Min. pt.

Thus **minimum value** $= f(\sqrt{2}, -\sqrt{2})$ **or** $f(-\sqrt{2}, \sqrt{2}) = -8$

77. *Find the extreme values of* $f(x, y) = x^3 y^2 (1 - x - y)$

\>> $\quad f(x, y) = x^3 y^2 - x^4 y^2 - x^3 y^3$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$

$$f_x = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3, \qquad f_y = 2x^3 y - 2x^4 y - 3x^3 y^2$$

Consider $f_x = 0$ and $f_y = 0$

ie., $\quad x^2 y^2 (3 - 4x - 3y) = 0$ **and** $x^3 y (2 - 2x - 3y) = 0$

$\Rightarrow \quad x = 0, \; y = 0, \; 4x + 3y = 3$ and $x = 0, \; y = 0, \; 2x + 3y = 2$

Let us form the pair of equations

$$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\} \left.\begin{array}{l} x = 0 \\ 2x + 3y = 2 \end{array}\right\} \left.\begin{array}{l} y = 0 \\ 2x + 3y = 2 \end{array}\right\} \left.\begin{array}{l} 4x + 3y = 3 \\ x = 0 \end{array}\right\} \left.\begin{array}{l} 4x + 3y = 3 \\ y = 0 \end{array}\right\} \left.\begin{array}{l} 4x + 3y = 3 \\ 2x + 3y = 2 \end{array}\right\}$$

The stationary points are $(0, 0)$ $(0, 2/3)$, $(1, 0)$ $(0, 1)$ $(3/4, 0)$ and $(1/2, 1/3)$

Also $\quad A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1 - 2x - y)$

$$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6 - 8x - 9y)$$

$$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y \qquad = 2x^3(1 - x - 3y)$$

It is evident that either $A = 0$ or $C = 0$ or both A and C are zero in respect of all the stationary points except $(1/2, 1/3)$

When $A = 0$ or $C = 0$, $AC - B^2 < 0$ and we shall examine the nature of the point $(1/2, 1/3)$. At this point we get
$A = -1/9$, $B = -1/12$ $\quad C = -1/8$

$\therefore \qquad AC - B^2 = 1/72 - 1/144 = 1/144 > 0$ But $A = -1/9 < 0$

Hence $(1/2, 1/3)$ is a maximum point.

Thus **maximum value of** $f(x, y) = f(1/2, 1/3) =$ **1/432**

78. *Show that* $z(x, y) = x^3 + y^3 - 3xy + 1$ *is minimum at* $(1, 1)$

>> $\qquad z(x, y) = x^3 + y^3 - 3xy + 1$

$$z_x = 3x^2 - 3y \ ; \ z_y = 3y^2 - 3x$$

Let $\quad A = z_{xx}$, $B = z_{xy}$, $C = z_{yy}$ $\quad \therefore \ A = 6x$, $B = -3$, $C = 6y$

Now, at $(1, 1)$ $z_x = 0$ and $z_y = 0$

Also $A = 6$, $B = -3$, $C = 6$ $\quad \therefore \quad AC - B^2 = 27 > 0$

Now at $(1, 1)$, $z_x = 0$ and $z_y = 0$ $AC - B^2 > 0$, $A = 6 > 0$

$\therefore \quad z(x, y)$ at $(1, 1)$ satisfy the necessary and suffecient conditions for minima

Thus **$z(x, y)$ is minimum at $(1, 1)$**

79. *Examine the function* $f(x, y) = 1 + \sin(x^2 + y^2)$ *for extremum*

>> $\qquad f(x, y) = 1 + \sin(x^2 + y^2)$

$$f_x = 2x\cos(x^2 + y^2), \ f_y = 2y\cos(x^2 + y^2)$$

We shall find points such that $f_x = 0$, $f_y = 0$

ie., $\qquad 2x\cos(x^2 + y^2) = 0$ and $2y\cos(x^2 + y^2) = 0$

$\therefore \quad x = 0, \ y = 0$ and $(0, 0)$ is the stationary point.

$$A = f_{xx} = -4x^2 \sin(x^2+y^2) + 2\cos(x^2+y^2)$$

$$B = f_{xy} = -4xy \sin(x^2+y^2)$$

$$C = f_{yy} = -4y^2 \sin(x^2+y^2) + 2\cos(x^2+y^2)$$

At $(0, 0) : A = 2, \ B = 0, \ C = 2 \quad \therefore \quad AC - B^2 = 4 > 0$

Since $AC - B^2 > 0$, $A = 2 > 0$, $(0, 0)$ is a minimum point and the **minimum value** of $f(x, y) = f(0, 0) = 1$

80. *Examine the function* $\sin x + \sin y + \sin(x+y)$ *for extreme values*

>> $f(x, y) = \sin x + \sin y + \sin(x+y)$

$f_x = \cos x + \cos(x+y) \ ; \ f_y = \cos y + \cos(x+y)$

Let us consider $f_x = 0$ and $f_y = 0$

ie., $\cos x + \cos(x+y) = 0$ and $\cos y + \cos(x+y) = 0$

or $\cos(x+y) = -\cos x$ and $\cos(x+y) = -\cos y$

Thus $-\cos x = -\cos y \quad \Rightarrow \quad x = y$

Putting $y = x$ in $\cos x + \cos(x+y) = 0$ we get $\cos x + \cos 2x = 0$

ie., $\cos x + 2\cos^2 x - 1 = 0$

ie., $2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \quad \cos x = 1/2, \ \cos x = -1 \quad \therefore \ x = \pi/3$ and $x = \pi$

Thus $(\pi/3, \pi/3)$ and (π, π) are the stationary points.

Let $A = f_{xx}, \ B = f_{xy}, \ C = f_{yy}$

ie., $A = -\sin x - \sin(x+y), \ B = -\sin(x+y), \ C = -\sin y - \sin(x+y)$

At $(\pi/3, \pi/3) : A = -\sin(\pi/3) - \sin(2\pi/3) = -(\sqrt{3}/2) - (\sqrt{3}/2)$

$$A = -\sqrt{3}, \ B = -(\sqrt{3}/2), \ C = -\sqrt{3}$$

$\therefore \quad AC - B^2 = 3 - (3/4) = 9/4 > 0 \ ; \ A = -\sqrt{3} < 0$

Hence $f(x, y)$ is maximum at $(\pi/3, \pi/3)$ and the maximum value is $\sin(\pi/3) + \sin(\pi/3) + \sin(2\pi/3) = \sqrt{3}/2 + \sqrt{3}/2 + \sqrt{3}/2$

Thus **maximum value = $3\sqrt{3}/2$**

At (π, π), $A = 0$, $B = 0$, $C = 0$ and hence the case needs further investigation.

EXERCISES

Find the maximum and minimum values of the following functions.

1. $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$
2. $x^3 y^2 (12 - 3x - 4y)$
3. $x^3 y^2 (12 - x - y)$
4. $x^2 y (x + 2y - 4)$
5. $x^3 + y^3 - 3axy, \quad a > 0$

ANSWERS

1. Min. value $= 0$ at $(2, 0)$ and max. value $= 4$ at $(0, 0)$
2. max value $= 8$ at $(2, 1)$
3. Max value $= 6922$ at $(6, 4)$
4. Min. value $= -2$ at $(2, 1/2)$
5. Min value $= -a^3$ at (a, a)

3.5 Errors and Approximations

3.51 Introduction

Many practical situations like computation involving various numerical quantities, taking measurements or readings, changing the scale etc. are susceptible to errors of various kinds based on various factors. Obviously it is going to have an impact on the final result which sometimes may be negligible or significant too. Further many numerical calculations will force us to approximate and in such cases any computed result will contain errors.

This topic gives an insight to such situations only, though the topic can be discussed still widely concerning the types of errors, error minimization etc.

3.52 Absolute, relative and percentage errors

We know that if y is dependent on x, that is if $y = f(x)$ then

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{lt}} \frac{\delta y}{\delta x}$$

This is equivalent to saying that the quantities $\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$ are equal if δx is small enough

$$\therefore \quad \delta y = \frac{dy}{dx}\,\delta x$$

$$\text{or} \quad \delta y = f'(x)\,\delta x \qquad \ldots (1)$$

The value δy is called the error in y due to an error δx in x.

Further in the computation of a numerical quantity, if x_0 is taken as an approximate value of x, that is when x is approximated to x_0

then $\delta x = |x - x_0|$ is called as the *absolute error in x.*

$(\delta x / x)$ is called as the *relative error in x.*

$(\delta x / x) \times 100$ is called as the *percentage error in x.*

Also if $z = f(x, y)$ then we have,

$$\delta z = \frac{\partial z}{\partial x}\,\delta x + \frac{\partial z}{\partial y}\,\delta y$$

The associated relative and percentage errors are respectively $(\delta z / z)$ and $(\delta z / z) \times 100$.

- We prefer to take logarithms first for the given expression and then take the differential as it would always give terms of the form $(\delta x / x)$
- We can as well multiply with 100 if the given data involves percentage errors.
- The following formulae will be useful.

1. Circle $[x^2 + y^2 = r^2] \ldots$ Area $= \pi r^2$
2. Ellipse $[x^2/a^2 + y^2/b^2 = 1] \ldots$ Area $= \pi ab$

Name of the Solid	Volume	Lateral or Curved Surface Area	Total Surface Area
1. Cube	a^3	$4a^2$	$6a^2$
2. Cuboid	lbh	$2(l+b)h$	$2(lb+bh+lh)$
3. Cylinder	$\pi r^2 h$	$2\pi rh$	$2\pi r(r+h)$
4. Cone	$1/3 \cdot \pi r^2 h$	πrl	$\pi r(r+l)$
5. Sphere	$4/3 \cdot \pi r^3$	-	$4\pi r^2$

where in the case of cone l is the slant height connected by the relation $l^2 = r^2 + h^2$

Note : The following results similar to that of differentials will also be useful.

(i) $\delta(cu) = c\,\delta u$ (ii) $\delta(u \pm v) = \delta u \pm \delta v$

(iii) $\delta(uv) = u\,\delta v + v\,\delta u$ (iv) $\delta\left(\dfrac{u}{v}\right) = \dfrac{v\,\delta u - u\,\delta v}{v^2}$

>> $PV^2 = K$ by data. Also $\dfrac{\delta P}{P} = 0.05$ and $\dfrac{\delta V}{V} = 0.025$

$\Rightarrow \quad \log P + 2\log V = \log K$

$\Rightarrow \quad \delta(\log P) + 2\delta(\log V) = \delta(\log K)$

ie., $\quad \dfrac{1}{P}\,\delta P + 2 \cdot \dfrac{1}{V}\,\delta V = \dfrac{1}{K}\,\delta K$

ie., $\quad 0.05 + 2(0.025) = \dfrac{\delta K}{K}$ or $\dfrac{\delta K}{K} = 0.1$

$\therefore \quad \dfrac{\delta K}{K} \times 100 = (0.1) \times 100 = 10$

Thus the error in K is 10%.

>> (i) $T = 2\pi\sqrt{l/g}$, g = constant, $\dfrac{\delta l}{l} \times 100 = 3$

$\Rightarrow \quad \log T = \log 2\pi + \dfrac{1}{2}(\log l - \log g)$

$\Rightarrow \quad \delta(\log T) = \delta(\log 2\pi) + \dfrac{1}{2}\delta(\log l) - \dfrac{1}{2}\delta(\log g)$

ie., $\quad \dfrac{\delta T}{T} = 0 + \dfrac{1}{2}\,\dfrac{\delta l}{l} - 0$

or $\quad \dfrac{\delta T}{T} \times 100 = \dfrac{1}{2}\left(\dfrac{\delta l}{l} \times 100\right) = \dfrac{1}{2}(3) = 1.5$

Thus the error in $T = 1.5\%$.

(ii) If g is not a constant we have,

$$\frac{\delta T}{T} \times 100 = \frac{1}{2}\left(\frac{\delta l}{l} \times 100\right) - \frac{1}{2}\left(\frac{\delta g}{g} \times 100\right)$$

The error in T will be **maximum** if the error in l is positive and the error in g is negative (or *vice-versa*) as the difference in errors converts into a sum.

$\therefore \quad \max\left(\frac{\delta T}{T} \times 100\right) = \frac{1}{2}(+1) - \frac{1}{2}(-3) = 2$

Thus **the maximum error in T is 2%.**

\>\> Consider $c = k \tan\theta$. k is taken as a constant.

$\Rightarrow \quad \log c = \log k + \log(\tan\theta)$

$\Rightarrow \quad \delta(\log c) = \delta(\log k) + \delta \log(\tan\theta)$

ie., $\quad \frac{1}{c}\,\delta c = 0 + \frac{\sec^2\theta}{\tan\theta}\,\delta\theta$

ie., $\quad \frac{\delta c}{c} = \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\cos^2\theta}\,\delta\theta$ or $\frac{\delta c}{c} = \frac{\delta\theta}{\sin\theta\cos\theta}$

ie., $\quad \frac{\delta c}{c} = \frac{2}{\sin 2\theta}\,\delta\theta$

The relative error in c being $\delta c/c$ is minimum when the denominator of the R.H.S is maximum and the maximum value of a sine function is 1.

$\therefore \quad \sin 2\theta = 1 \Rightarrow 2\theta = 90^0$ or $\theta = 45^0$.

Thus the relative error in c is minimum when $\theta = 45^o$

84. *If $T = \frac{1}{2}mv^2$ is the kinetic energy, find approximately the change in T as m changes from 49 to 49.5 and v changes from 1600 to 1590.*

\>\> We have by data $T = \frac{1}{2}mv^2$ and

$m = 49,\ m + \delta m = 49.5 \ \therefore\ \delta m = 0.5$

$v = 1600,\ v + \delta v = 1590 \ \therefore\ \delta v = -10$

We have to find δT. (*logarithm is not required*)

$$\therefore \quad \delta T = \frac{1}{2}\,\delta\,(mv^2)$$

$$= \frac{1}{2}\left[\,m\,(2v\,\delta v) + \delta m \cdot v^2\,\right]$$

$$i.e., \quad = \frac{1}{2}\left[\,(49)\,(2)\,(1600)\,(-10) + (0.5)\,(1600)^2\,\right] = -1{,}44{,}000$$

Thus the change in $T = \delta T = -1{,}44{,}000$.

>> $p\,v^{1.4}$ = constant = c (*say*), by data.

$$\Rightarrow \quad \log p + 1.4 \log v = \log c$$

$$\Rightarrow \quad \delta(\log p) + 1.4\,\delta(\log v) = \delta(\log c)$$

$$i.e., \quad \frac{\delta p}{p} + 1.4\left(\frac{\delta v}{v}\right) = 0, \text{ But } \frac{\delta v}{v} \times 100 = -\frac{1}{2}, \text{ by data.}$$

$$\therefore \quad \frac{\delta p}{p} \times 100 + 1.4\left(\frac{\delta v}{v} \times 100\right) = 0 \quad \text{or} \quad \frac{\delta p}{p} \times 100 = +0.7$$

Thus the percentage increase in pressure = 0.7

>> Let x be the deflection and we have by data

$$x \propto \frac{wl^3}{d^4} \Rightarrow x = k\,\frac{wl^3}{d^4},\ k \text{ being a constant.}$$

$$\therefore \quad \log x = \log k + \log w + 3 \log l - 4 \log d$$

$$\Rightarrow \quad \delta(\log x) = \delta(\log k) + \delta(\log w) + 3\delta(\log l) - 4\delta(\log d)$$

$$\Rightarrow \quad \frac{\delta x}{x} = 0 + \frac{\delta w}{w} + 3\left(\frac{\delta l}{l}\right) - 4\left(\frac{\delta d}{d}\right)$$

$$\Rightarrow \quad \frac{\delta x}{x} \times 100 = \frac{\delta w}{w} \times 100 + 3\left(\frac{\delta l}{l} \times 100\right) - 4\left(\frac{\delta d}{d} \times 100\right)$$

But $\quad \dfrac{\delta w}{w} \times 100 = 5,\ \dfrac{\delta l}{l} \times 100 = 4,\ \dfrac{\delta d}{d} \times 100 = 3$, by data.

$$\therefore \quad \frac{\delta x}{x} \times 100 = 5 + 3(4) - 4(3) = 5$$

Thus the percentage increase in the deflection = 5

$>>$ $\quad$ By data, $\dfrac{2}{f} = \dfrac{1}{v} - \dfrac{1}{u}$ and $\delta u = \delta v = e$

$$\Rightarrow \quad 2\delta\left(\frac{1}{f}\right) = \delta\left(\frac{1}{v}\right) - \delta\left(\frac{1}{u}\right)$$

$$ie, \quad 2\left(-\frac{1}{f^2}\,\delta f\right) = -\frac{1}{v^2}\,\delta v + \frac{1}{u^2}\,\delta u$$

$$ie., \quad \frac{1}{f}\,\delta f = \frac{f}{2}\left[\frac{1}{v^2}\,\delta v - \frac{1}{u^2}\,\delta u\right]$$

$$ie., \quad \frac{\delta f}{f} = e \cdot \frac{f}{2}\left(\frac{1}{v} + \frac{1}{u}\right)\left(\frac{1}{v} - \frac{1}{u}\right)$$

$$ie, \quad \frac{\delta f}{f} = e \cdot \frac{f}{2}\left(\frac{1}{v} + \frac{1}{u}\right)\frac{2}{f}$$

Thus the relative error in $f = e\left(\dfrac{1}{u} + \dfrac{1}{v}\right)$

$>>$ $\quad \dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{q}\ ;\ p = q = 20\ ;\ \delta p = \delta q = 0.5$

$$\Rightarrow \quad \delta\left(\frac{1}{f}\right) = \delta\left(\frac{1}{p}\right) + \delta\left(\frac{1}{q}\right)$$

$$ie., \quad -\frac{1}{f^2}\,\delta f = -\frac{1}{p^2}\,\delta p - \frac{1}{q^2}\,\delta q$$

or $\quad \delta f = f^2 \left[\dfrac{1}{p^2}\, \delta p + \dfrac{1}{q^2}\, \delta q \right] = \dfrac{2f^2}{p^2}\, \delta p \quad \because\ p = q,\ \delta p = \delta q$

Also $\quad \dfrac{1}{f} = \dfrac{1}{20} + \dfrac{1}{20} = 0.1 \quad$ or $\quad f = 10$

$$\therefore \quad \delta f = \frac{2\,(10)^2}{(20)^2}\,(0.5) = 0.25$$

Thus the maximum error in $f = 0.25$

>> If l and b denotes the length and the breadth of a rectangle then the area (A) is given by

$A = lb$; Also by data $\dfrac{\delta l}{l} \times 100 = 1 = \dfrac{\delta b}{b} \times 100$

We have $\quad \log A = \log l + \log b$

$\Rightarrow \quad \delta(\log A) = \delta(\log l) + \delta(\log b)$

ie., $\quad \dfrac{\delta A}{A} = \dfrac{\delta l}{l} + \dfrac{\delta b}{b}$ or $\dfrac{\delta A}{A} \times 100 = \dfrac{\delta l}{l} \times 100 + \dfrac{\delta b}{b} \times 100$

$\therefore \quad \dfrac{\delta A}{A} \times 100 = 1 + 1 = 2$

Thus the error in the area is 2%

>> For the ellipse $x^2/a^2 + y^2/b^2 = 1$ the area (A) is given by πab where $2a$ and $2b$ are the lengths of the major and minor axis.

Let $2a = x$ and $2b = y$.

By data $\dfrac{\delta x}{x} \times 100 = 1,\ \dfrac{\delta y}{y} \times 100 = 1.$

$$A = \pi ab = \pi \cdot \frac{x}{2} \cdot \frac{y}{2} = \frac{\pi}{4}\, xy$$

$\therefore \quad \log A = \log(\pi/4) + \log x + \log y$

$\Rightarrow \quad \delta(\log A) = \delta \log(\pi/4) + \delta(\log x) + \delta(\log y)$

ie, $\quad \dfrac{\delta A}{A} = 0 + \dfrac{\delta x}{x} + \dfrac{\delta y}{y}$ or $\dfrac{\delta A}{A} \times 100 = \dfrac{\delta x}{x} \times 100 + \dfrac{\delta y}{y} \times 100$

$\therefore \quad \dfrac{\delta A}{A} \times 100 = 1 + 1 = 2$

Thus error in the area = 2%

91. In [illegible] errors of 2% and 1% are made in height and radius of the base [illegible]ectively. Find the percentage error in th[illegible]

\>> We know the volume V of a right circular cone is given by

$V = \dfrac{1}{3}\pi r^2 h$. By data $\dfrac{\delta h}{h} \times 100 = 2, \ \dfrac{\delta r}{r} \times 100 = 1$

$\Rightarrow \quad \log V = \log(\pi/3) + 2\log r + \log h$

$\Rightarrow \quad \delta(\log V) = \delta\log(\pi/3) + 2\delta(\log r) + \delta(\log h)$

ie, $\quad \dfrac{\delta V}{V} = 2\dfrac{\delta r}{r} + \dfrac{\delta h}{h}$ or $\dfrac{\delta V}{V} \times 100 = 2\dfrac{\delta r}{r} \times 100 + \dfrac{\delta h}{h} \times 100$

$\therefore \quad \dfrac{\delta V}{V} \times 100 = 2(1) + 2 = 4$

Thus the error in the volume = 4%

[illegible]

\>> The volume of the sphere is given by $V = \dfrac{4}{3}\pi r^3$

$\Rightarrow \quad \log V = \log(4\pi/3) + 3\log r \ ; \ \dfrac{\delta r}{r} \times 100 = 1 \cdot 25$ by data

$\Rightarrow \quad \delta(\log V) = 0 + 3\delta(\log r)$

ie., $\quad \dfrac{\delta V}{V} = 3 \cdot \dfrac{\delta r}{r}$ or $\dfrac{\delta V}{V} \times 100 = 3 \cdot \dfrac{\delta r}{r} \times 100$

$\therefore \quad \dfrac{\delta V}{V} \times 100 = 3(1.25) = 3.75$

Thus the error in the volume = 3.75%

The surface area of the sphere $(S) = 4\pi r^2$

$\Rightarrow \quad \log S = \log(4\pi) + 2\log r$

$\Rightarrow \quad \delta(\log S) = 2\delta(\log r)$

ie., $\dfrac{\delta S}{S} = 2 \cdot \dfrac{\delta r}{r}$ or $\dfrac{\delta S}{S} \times 100 = 2\,\dfrac{\delta r}{r} \times 100$

$$\therefore \quad \frac{\delta S}{S} \times 100 = 2\,(1.25) = 2.5$$

Thus the error in the surface area = 2.5%

>> The volume (V) of a right circular cylinder is given by $V = \pi r^2 h$. If D is the diameter then $r = D/2$

$$\therefore \quad V = \pi\,\frac{D^2}{4} \cdot h \quad ie., \quad V = \frac{\pi}{4}\,D^2 h$$

By data, $D = 4.5,\ \delta D = 0.1\ ,\ h = 8.25,\ \delta h = 0.1$

Now $\quad \delta V = \dfrac{\pi}{4}\,\delta\,(D^2 h)$

ie., $\quad \delta V = \dfrac{\pi}{4}\left(D^2 \cdot \delta h + 2D\,\delta D \cdot h\right)$

$$= \frac{\pi}{4}\{(4.5)^2(0.1) + 2(4.5)(0.1)(8.25)\} = 7.42$$

Thus the error in the volume = 7.42cm^3

Also the lateral surface area (S) of the cylinder is given by $S = 2\pi rh$ where $r = D/2$

ie., $\quad S = \pi Dh$

$$\therefore \quad \delta S = \pi\,(D\,\delta h + h\,\delta D)$$

$$= \pi\left(4.5 \times 0.1 + 8.25 \times 0.1\right)$$

ie., $\quad \delta S = (0.1)\,\pi\,(12.75) = 4.0055 \approx 4$

Thus the error in the surface area = 4cm^2

94. [illegible]

>> Volume of the sphere $(V) = \frac{4}{3}\pi r^3$

$$\delta V = \frac{4\pi}{3}\delta(r^3) = \frac{4\pi}{3}(3r^2\,\delta r) = 4\pi r^2\,\delta r$$

The relative error in the volume :

$$\frac{\delta V}{V} = \frac{4\pi r^2\,\delta r}{4/3\cdot\pi r^3} = 3\,\frac{\delta r}{r}. \quad \text{But } r = 10,\ \delta r = 0.02.$$

$$\frac{\delta V}{V} = \frac{3\,(0.02)}{10} = 0.006$$

Thus the relative error in the volume = 0.006 cm^3

95. [illegible]

>> By data $r = 6,\ h = 18,\ \delta r = \delta h = 0.15$

We have $V = \pi r^2 h$, being the volume of the right circular cylinder

$$\Rightarrow \quad \log V = \log\pi + 2\log r + \log h$$

$$\Rightarrow \quad \delta(\log V) = \delta(\log\pi) + 2\delta(\log r) + \delta(\log h)$$

ie., $$\frac{\delta V}{V} = 0 + 2\,\frac{\delta r}{r} + \frac{\delta h}{h} = \delta r\left(\frac{2}{r} + \frac{1}{h}\right) \quad \because\ \delta r = \delta h$$

$$\frac{\delta V}{V} = 0.15\left[\frac{2}{6} + \frac{1}{18}\right] = 0.15\left(\frac{7}{18}\right) = 0.0583$$

$$\Rightarrow \quad \frac{\delta V}{V}\times 100 = 5.83$$

Thus the percentage error in the volume = 5.83

We have $S = 2\pi r h$, being the surface area of the cylinder.

$$\Rightarrow \quad \log S = \log(2\pi) + \log r + \log h$$

$$\Rightarrow \quad \delta(\log S) = 0 + \delta(\log r) + \delta(\log h)$$

ie., $$\frac{\delta S}{S} = \frac{\delta r}{r} + \frac{\delta h}{h} = \delta r\left(\frac{1}{r} + \frac{1}{h}\right) = 0.15\left(\frac{1}{6} + \frac{1}{18}\right) = 0.0333$$

$$\therefore \quad \frac{\delta S}{S} \times 100 = 3.33$$

Thus the percentage error in the surface area = 3.33

[illegible]
bricks cost Rs 1,100 per thousand find the approximate error in the cost

>> Let $l,\ b,\ h$ respectively be the length, breadth and height of the pile of bricks so that the volume (V) of the pile is

$$V = lbh$$

$$\Rightarrow \quad \log V = \log l + \log b + \log h$$

$$\Rightarrow \quad \delta(\log V) = \delta(\log l) + \delta(\log b) + \delta(\log h)$$

ie., $$\frac{\delta V}{V} = \frac{\delta l}{l} + \frac{\delta b}{b} + \frac{\delta h}{h} \qquad \ldots (1)$$

Since the tape is stretched by 1%, the error in $l,\ b,\ h$ is 1%

ie., $$\frac{\delta l}{l} \times 100 = \frac{\delta b}{b} \times 100 = \frac{\delta h}{h} \times 100 = 1$$

or $$\frac{\delta l}{l} = \frac{\delta b}{b} = \frac{\delta h}{h} = \frac{1}{100} = 0.01$$

(1) becomes, $$\frac{\delta V}{V} = 0.01 + 0.01 + 0.01 = 0.03 \qquad \ldots (2)$$

But we have by data $l = 2,\ b = 15,\ h = 1.2$

$$\therefore \quad V = lbh = 2 \times 15 \times 1.2 = 36 \text{ cubic metres.}$$

Hence (2) gives, $\delta V = 0.03 \times V = 0.03 \times 36 = 1.08$ cubic metres
$\therefore$ associated number of bricks in $\delta V = 450 \times 1.08 = 486$

But the cost of bricks = Rs.1100 per thousand = Rs 1.10 per brick.

$\therefore$ the approximate error in the cost = $486 \times 1.10 =$ Rs.534.6

Thus the approximate error in the cost is Rs.534.60

97. *If the sides and angles of a triangle ABC vary in such a way that the circum radius remains constant, prove that*

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{c\ldots} \text{[illegible]}$$

>> If the triangle ABC is inscribed in a circle of radius r and if a, b, c respectively denotes the sides opposite to the angles A, B, C we have the sine rule (*formula*) given by

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

or $\quad a = 2r \sin A,\ b = 2r \sin B,\ c = 2r \sin C$

$\Rightarrow \quad \delta a = 2r\,\delta(\sin A),\quad \delta b = 2r\,\delta(\sin B),\quad \delta c = 2r\,\delta(\sin C)$

ie., $\quad \delta a = 2r \cos A\,\delta A,\quad \delta b = 2r \cos B\,\delta B,\quad \delta c = 2r \cos C\,\delta C$

or $\quad \dfrac{\delta a}{\cos A} = 2r\,\delta A\,,\ \dfrac{\delta b}{\cos B} = 2r\,\delta B\,,\ \dfrac{\delta c}{\cos C} = 2r\,\delta C$

Adding all these results we get,

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2r(\delta A + \delta B + \delta C) = 2r\,\delta(A + B + C)$$

But $\quad A + B + C = 180 = \pi$ radians $=$ constant.

$\Rightarrow \quad \delta(A + B + C) = \delta(\text{constant}) = 0$

Thus $\quad \dfrac{\boldsymbol{\delta a}}{\mathbf{cos}\,\boldsymbol{A}} + \dfrac{\boldsymbol{\delta b}}{\mathbf{cos}\,\boldsymbol{B}} + \dfrac{\boldsymbol{\delta c}}{\mathbf{cos}\,\boldsymbol{C}} = 0$

98. [illegible] 30 [illegible] 65' [illegible] *in the computation of the length of the third side*

>> Given two sides and the included angle, to compute the thrid side we have to use the cosine formula in the standard form :

$$a^2 = b^2 + c^2 - 2bc \cos A$$

By data, $b = 60,\ c = 30,\ \hat{A} = 65^0\,24'$

But $\quad 1^\circ = 60' \Rightarrow 24' = \dfrac{24}{60} = 0.4^\circ \quad \therefore\ \hat{A} = 65.4^\circ$

Now $\quad a^2 = 60^2 + 30^2 - 2(60)(30)\cos 65.4^\circ = 3001.389148$

$\qquad a = 54.785$

Now by taking $\hat{A} = 65^{\circ}$ we get,

$$a^2 = 60^2 + 30^2 - 2\,(60)\,(30)\cos 65^{\circ} = 2978.574258$$

$\therefore$ the new $a = a_0\ (say) = 54.576$

Hence the absolute error in $'a' = |\,a - a_0\,| = 0.209 \approx$ **0.21**

due to a fluctuation of 25 minutes in the measurement.

>> Let x denote the area of the triangle ABC.

$$x = \frac{1}{2}\,bc\sin A, \text{ by data.}$$

$$\Rightarrow \quad \log x = \log(1/2) + \log b + \log c + \log(\sin A)$$

$$\Rightarrow \quad \delta(\log x) = \delta[\log(1/2)] + \delta(\log b) + \delta(\log c) + \delta\log(\sin A)$$

ie., $$\frac{\delta x}{x} = 0 + 0 + 0 + \frac{1}{\sin A}\cos A\,\delta A = \cot A\,\delta A$$

ie., $$\frac{\delta x}{x} = \cot A\ \delta A \qquad \ldots (1)$$

$$\delta A = 25' \text{ by data. or } \delta A = \frac{25'}{60} = \frac{5^{0}}{12} = \frac{5}{12}\cdot\frac{\pi}{180} \text{ radians}$$

ie., $$\delta A = \frac{\pi}{432} \text{ radians.}$$

Substituting this value in (1) we get,

$$\frac{\delta x}{x} = \cot 52^{\circ}\cdot\frac{\pi}{432} \quad \text{or} \quad \frac{\delta x}{x}\times 100 = \cot 52^{\circ}\cdot\frac{\pi}{432}\times 100$$

$$\therefore \quad \frac{\delta x}{x}\times 100 = 0.5682 \approx 0.57$$

Thus the percentage error in the area = 0.57

>> Area of the circle $(A) = \pi r^2$

$\Rightarrow \quad \log A = \log \pi + 2 \log r$

$\Rightarrow \quad \delta(\log A) = 0 + 2\delta(\log r)$ *ie.,* $\dfrac{\delta A}{A} = 2\dfrac{\delta r}{r}$

But $\quad \delta r = 1$ mm by data or $\delta r = 0.1$ cm and $r = 50$

$\therefore \quad \dfrac{\delta A}{A} = \dfrac{2 \times 0.1}{50} = 0.004$

or $\quad \dfrac{\delta A}{A} \times 100 = 0.004 \times 100 = 0.4$

Hence the percentage error in the area = 0.4

EXERCISES

1. Find the percentage error in calculating the area of a rectangle when an error of $+0.5\,\%$ is made while measuring the sides.

2. Find the percentage error in calculating the area of an ellipse when an error of -3% is made while measuring the major axis and an error of 2% is made while measuring the minor axis.

3. Find the percentage error in the volume of a right circular cone when an error of 2% is committed while measuring its height as well as the radius of its base

4. Find the percentage error in calculating the volume and the surface area of the sphere due to an error of 0.75% in the radius of the sphere

5. The diameter and the altitude of a right circular cylinder are measured as 4 cms and 6 cms respectively If the possible error in each of the measurements is 0.1 cms, show that the error in the volume and the lateral surface are respectively $1.6\,\pi$ and π.

6. Find the percentage error in calculating the area of a triangle due to an error of 2% and 1% respectively in the base and the altitude.

7. In the relation $PV = RT$, find approximately the change in P correct to two decimal places as T changes from 500 to 503, V from 15.20 to 15.25 given that $P = 4000$ and R is a constant.

8. If $T = 1/2\ mv^2$, find approximately the changes in T as m changes to 50.5 from 50 and V to 1495 from 1500.

9. In a triangle ABC the side BC and the angle opposite to it remain constant and the other sides and angles vary slightly, show that

$$\frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0.$$

10. A triangle ABC has $AB = 70$ cms, $BC = 40$ cms, $\hat{B} = 64^\circ\, 12'$. Find the error in the computation of the side AC by approximating $\hat{B}$ to 65°.

ANSWERS

1. 1	2. -1	3. 6	4. 2.25, 1.5
6. 3	7. 10.85	8. 187500	10. 0.55

Unit - IV

VECTOR CALCULUS

4.1 | Vector Differentiation

4.11 | Introduction

Basically *Vector* is a quantity having both magnitude and direction. Vector quantities like force, velocity, acceleration etc. have lot of reference in physical and engineering problems. We are familiar with vector algebra which gives an exposure to all the basic concepts related to vectors.

Differentiation and Integration are well acquainted topics in calculus. In the background of all these we discuss this chapter *Vector Calculus* comprising *Vector Differentiation*. Many concepts are highly significant in various branches of engineering.

4.12 | Basic Concepts : Vector function of a single variable and the derivative of a vector

Let the position vector of a point $P(x, y, z)$ in space be

$$\vec{r} = x\,i + y\,j + z\,k$$

If x, y, z are all functions of a single parameter t, then $\vec{r}$ is said to be a vector function of t which is also referred to as a *vector point function* usually denoted as $\vec{r} = \vec{r}(t)$. As the parameter t varies, the point P traces a curve in space. Therefore

$$\vec{r} = x(t)\,i + y(t)\,j + z(t)\,k$$

is called as the vector equation of the curve.

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \frac{dx}{dt}\,i + \frac{dy}{dt}\,j + \frac{dz}{dt}\,k$$

is a **vector along the tangent to the curve at** P

If t is the time variable,

$$\vec{v} = \frac{d\vec{r}}{dt}$$ gives the **velocity of the particle at time** t.

Further $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity $\vec{v}$ and is called the **acceleration of the particle at time t**

Properties

1. $\frac{d}{dt}\left[c_1\vec{r}_1(t) \pm c_2\vec{r}_2(t)\right] = c_1\vec{r}_1{}'(t) \pm c_2\vec{r}_2{}'(t)$ where c_1, c_2 are constants.

2. $\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \vec{F}\cdot\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\cdot\vec{G}$

3. $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\times\vec{G}$

where $\vec{F} = \vec{F}(t)$ and $\vec{G} = \vec{G}(t)$

3. [illegible] and Vector point functions

If to every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a ***scalar point function*** and we say that a ***scalar field*** ϕ is defined in R

Examples : **1.** $\phi = x^2 + y^2 + z^2$ **2.** $\phi = xy^2z^3$

If to every point (x, y, z) of a region R in space there corresponds a vector $\vec{A}(x, y, z)$ then $\vec{A}$ is called a ***vector point function*** and we say that a ***vector field*** $\vec{A}$ is defined in R.

Examples : **1.** $\vec{A} = x^2 i + y^2 j + z^2 k$ **2.** $\vec{A} = xyz\, i + yz\, j + z\, k$

Operators (i) The ***vector differential operator*** ∇, *read as "Nabla" or "Del"* is defined by

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$$

(ii) The ***Laplacian operator*** ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$$

We now proceed to define four important quantities associated with the operators ∇ and ∇^2.

4.14 Gradient, Divergence, Curl and Laplacian

Gradient of a scalar field

If $\phi(x, y, z)$ is a continuously differentiable scalar function then the **gradient of** ϕ (grad ϕ *in precise*) is defined to be $\nabla\phi$.

ie., $$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

Obviously $\nabla\phi$ is a vector quantity.

Divergence of a vector field

If $\vec{A}(x, y, z)$ is a continuously differentiable vector function then **divergence of** $\vec{A}$ (div $\vec{A}$ *in precise*) is defined to be $\nabla \cdot \vec{A}$

If $\vec{A} = a_1 i + a_2 j + a_3 k$, where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (a_1 i + a_2 j + a_3 k)$$

ie., $$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

Clearly div $\vec{A}$ is a scalar quantity.

Curl of a vector field

If $\vec{A}(x, y, z)$ is a continuously differentiable vector function then **curl of** $\vec{A}$ (curl $\vec{A}$ *in precise*) is defined to be $\nabla \times \vec{A}$

If $\vec{A} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z} \right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)$$

Clearly curl $\vec{A}$ is a vector quantity.

Laplacian If $\phi(x, y, z)$ is a continuously differentiable scalar function and $\vec{A}(x, y, z)$ is a continuously differentiable vector function we can define the Laplacian for ϕ as well as for $\vec{A}$ as follows.

$$\text{Laplacian of } \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\text{Laplacian of } \vec{A} = \nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$$

Note *If ϕ is a scalar function, the equation $\nabla^2 \phi = 0$ is called Laplace's equation and a function which satisfies Laplace's equation is called a harmonic function.*

Also $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$ *is called Laplace's equation in two dimensions.*

Obviously Laplacian of a scalar function is a scalar quantity and Laplacian of a vector function is a vector quantity.

Remark : **1.** *If $\phi(x, y, z)$ is a scalar function then we have*

$$\text{grad}\phi = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\text{div}(\text{grad}\,\phi) = \nabla \cdot \nabla\phi$$

ie.,

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi$$

Thus **div (grad ϕ) = $\nabla^2 \phi$ or $\nabla \cdot \nabla \phi = \nabla^2 \phi$**

2. *The computation of gradient, divergence, curl and laplacian is a combination of the concepts of vector algebra and partial differentiation.*

Here are a few illustrations.

1. Given $\phi = x^2 y + y^2 z + z^2 x$ let us find $\nabla \phi$ and $\nabla^2 \phi$

$$\nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

ie., $\nabla\phi = (2xy + z^2)\, i + (x^2 + 2yz)\, j + (y^2 + 2xz)\, k$

Next $\nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}$ or $\nabla \cdot (\nabla\phi)$

ie., $\nabla^2\phi = 2y + 2z + 2x = 2(x + y + z)$

2. Given $\vec{A} = x^2yz\,i + y^2zx\,j + z^2xy\,k$ let us find div $\vec{A}$, curl $\vec{A}$ and $\nabla^2\vec{A}$.

$$\text{div}\,\vec{A} = \nabla \cdot \vec{A}$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(x^2yz\,i + y^2zx\,j + z^2xy\,k\right)$$

$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(y^2zx) + \frac{\partial}{\partial z}(z^2xy)$$

$$= 2xyz + 2xyz + 2xyz = 6xyz \quad \therefore \quad \text{div}\,\vec{A} = 6xyz$$

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & y^2zx & z^2xy \end{vmatrix}$$

$$= i(z^2x - y^2x) - j(z^2y - x^2y) + k(y^2z - x^2z)$$

$$\text{curl}\,\vec{A} = x(z^2 - y^2)\,i + y(x^2 - z^2)\,j + z(y^2 - x^2)\,k$$

$$\nabla^2\vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2xyz\,i + y^2z\,j + z^2y\,k) + \frac{\partial}{\partial y}(x^2z\,i + 2xyz\,j + z^2x\,k)$$
$$+ \frac{\partial}{\partial z}(x^2y\,i + y^2x\,j + 2xyz\,k)$$

$$= 2yz\,i + 2xz\,j + 2xy\,k \quad \therefore \quad \nabla^2\vec{A} = 2\,(yzi + zxj + xyk)$$

Geometrical meaning of the gradient

Theorem *If $\phi(x, y, z)$ is a scalar function then $\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = c$*

Proof : Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on the surface $\phi(x, y, z) = c$. Also let

$$\vec{r} = x(t)\,i + y(t)\,j + z(t)\,k$$

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,i + \frac{dy}{dt}\,j + \frac{dz}{dt}\,k \text{ is tangential to the surface at } P$$

We have $\quad \nabla\phi = \frac{\partial\phi}{\partial x}\,i + \frac{\partial\phi}{\partial y}\,j + \frac{\partial\phi}{\partial z}\,k$

Taking the dot product of these two vectors we have

$$\nabla\phi \cdot \frac{d\vec{r}}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \qquad \ldots (1)$$

Also, let us consider $\phi(x, y, z) = c$ where $x = x(t)$, $y = y(t)$, $z = z(t)$ and c is a constant.

Differentiating w.r.t. t on both sides we have $\frac{d\phi}{dt} = 0$ and using the concept of the differentiation of composite functions (*Total derivative*) in L.H.S. we obtain

$$\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} = 0$$

or $\quad \nabla\phi \cdot \frac{d\vec{r}}{dt} = 0$ by using (1)

$\Rightarrow \quad \nabla\phi$ is perpendicular to $\frac{d\vec{r}}{dt}$

Since $\frac{d\vec{r}}{dt}$ is a vector tangential to the surface at P **we can conclude that $\nabla\phi$ is along the normal to the surface $\phi(x, y, z) = c$ at P.**

This proves the theorem.

Note . 1. *Obviously the unit vector normal $\hat{n}$ along $\nabla\phi$ is given by $\hat{n} = \nabla\phi / |\nabla\phi|$*

2. *The angle between the two surfaces is defined to be equal to the angle between their normals. If $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ be the equations of the two surfaces then*

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$, *where θ is the angle between the normals.*

If $\theta = \pi/2$ then the surfaces are said to intersect each other orthogonally.

When $\theta = \pi/2$, $\cos\theta = \cos(\pi/2) = 0 \Rightarrow \nabla\phi_1 \cdot \nabla\phi_2 = 0$

This is the condition for the surfaces to intersect at right angles.

Directional derivative

If $\phi(x, y, z)$ is a scalar function and $\vec{d}$ is a given direction then $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \vec{d} / |\vec{d}|$ is called as the *directional derivative* of ϕ along $\hat{n}$.

Theorem. *The directional derivative of a scalar function ...*

Proof. We have by definition $\nabla\phi \cdot \hat{n}$ is the directional derivative of ϕ along $\hat{n}$.

Now, by the definition of the dot product we have

$$\nabla\phi \cdot \hat{n} = |\nabla\phi|\,|\hat{n}|\cos\theta$$

where θ is the angle between $\nabla\phi$ and $\hat{n}$. Since $|\hat{n}| = 1$ we have,

$$\nabla\phi \cdot \hat{n} = |\nabla\phi| \cos\theta$$

$\cos\theta$ when $\theta = 0$ has the maximum value equal to 1. If $\theta = 0$, $\nabla\phi$ coincides with $\hat{n}$ or we can say that $\nabla\phi$ will be along $\hat{n}$.

∴ the directional derivative is maximum along $\nabla\phi$ and its maximum value is equal to $|\nabla\phi|$.

This proves the theorem.

Note *Maximum directional derivative of a scalar function ϕ at a point P is also called as the normal derivative of the scalar function at P. Normal derivative $= |\nabla\phi|$ at P*

Tangent plane and Normal line

If $\phi(x, y, z) = c$ be the equation of a surface and $P(x_1, y_1, z_1)$ is a point on it then the equation of the ***tangent plane*** at the point P is given by

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

Also the equation of the ***normal line*** is given by

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C} \quad \text{where we have}$$

$$A = \left(\frac{\partial\phi}{\partial x}\right)_{(x_1, y_1, z_1)}, \quad B = \left(\frac{\partial\phi}{\partial y}\right)_{(x_1, y_1, z_1)}, \quad C = \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)}$$

Physical meaning of divergence

If $\vec{V}(x, y, z)$ represents any physical quantity, the divergence of $\vec{V}$ gives the rate at which the physical quantity is originating at that point per unit volume.

An Illustration : Let us suppose that a fluid is moving such that its velocity at any point $P(x, y, z)$ is given by the vector point function $\vec{V}(x, y, z)$. Consider a small parallelopiped of volume $\delta x\, \delta y\, \delta z$ through which the fluid is passing.

If $\quad \vec{V}(x, y, z) = v_1 i + v_2 j + v_3 k$ then $\operatorname{div} \vec{V} = \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z}$ gives the total gain in the volume of the fluid per unit volume per unit time.

$\operatorname{div} \vec{V} = 0$ is called as the continuity equation of an incompressible fluid.

A vector $\vec{V}$ whose divergence is zero is called a "solenoidal vector".

Physical meaning of curl

Curl means rotation. **A vector function $\vec{V}(x, y, z)$ is said to be "irrotational" if curl $\vec{V} = \vec{0}$**

An illustration Let us suppose that a rigid body is rotating about a fixed axis through a point O. If $\vec{\omega}$ is the constant angular velocity and $\vec{v}$ is the velocity of a particle at

a point $P(x, y, z)$ of the body having the position vector $\vec{r}$ then we know that $\vec{v} = \vec{\omega} \times \vec{r}$. We can easily show that curl $\vec{v} = 2\vec{\omega}$ (*Refer Example 26*)

Thus curl of the velocity vector is equal to twice the angular velocity of rotation. This is an illustration to show that curl is analogous to "rotation". In general we can as well say that the curl of any vector point function will give the measure of the angular velocity at any point.

WORKED PROBLEMS

Important points to remember

1. $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$

   $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$

2. grad $\phi = \nabla\phi$; div $\vec{A} = \nabla \cdot \vec{A}$; curl $\vec{A} = \nabla \times \vec{A}$

   Laplacian of $\phi = \nabla^2\phi = \nabla \cdot \nabla\phi$

3. $\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = c$ and $\nabla\phi / |\nabla\phi|$ is the unit vector normal to the surface.

4. Directional derivative of $\phi(x, y, z)$ along a given direction $\vec{D}$ is $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \vec{D} / |\vec{D}|$ and also directional derivative is maximum along $\nabla\phi$

5. If $\phi(x, y, z)$ represents the temperature function then the directional derivative of ϕ along $\vec{D}$ is the rate of change of temperature along $\vec{D}$

Find the unit vector normal to the following surfaces at the indicated points

1. $x^2 y + 2xz = 4$ at $(2, -2, 3)$
2. $x y^3 z^2 = 4$ at $(-1, -1, 2)$
3. $x^2 y - 2xz + 2y^2 z^4 = 10$ at $(2, 1, -1)$

1. Let $\phi = x^2 y + 2xz$. $\nabla\phi$ is a vector normal to the surface.

We have $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

ie., $\nabla\phi = (2xy + 2z)\, i + x^2 j + 2x\, k$

$[\nabla\phi]_{(2,-2,3)} = -2i + 4j + 4k = 2(-i + 2j + 2k)$

The required unit vector normal $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Thus $\quad \hat{n} = \dfrac{2(-i+2j+2k)}{\sqrt{2^2(1+4+4)}} = \dfrac{-i+2j+2k}{3}$

---

2. Let $\phi = xy^3 z^2$. $\nabla\phi$ is a vector normal to the surface.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $\quad \nabla\phi = y^3 z^2\, i + 3xy^2 z^2\, j + 2xy^3 z\, k$

$$\left[\nabla\phi\right]_{(-1,\,-1,\,2)} = -4i - 12j + 4k = -4(i+3j-k)$$

The required unit vector normal $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Thus $\quad \hat{n} = \dfrac{-4(i+3j-k)}{\sqrt{4^2(1+9+1)}} = -\dfrac{(i+3j-k)}{\sqrt{11}}$

---

3. Let $\phi = x^2 y - 2xz + 2y^2 z^4$

$\nabla\phi$ is a vector normal to the surface

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

$$\nabla\phi = (2xy - 2z)\, i + (x^2 + 4yz^4)\, j + (-2x + 8y^2 z^3)\, k$$

$$[\nabla\phi]_{(2,\,1,\,-1)} = 6i + 8j - 12k = 2(3i + 4j - 6k)$$

The unit vector normal $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Thus $\quad \hat{n} = \dfrac{2(3i+4j-6k)}{\sqrt{2^2(9+16+36)}} = \dfrac{3i+4j-6k}{\sqrt{61}}$

Find the directional derivatives of the following

4. $\phi = x^2 yz + 4xz^2$ *at* $(1, -2, -1)$ *along* $2i - j - 2k$

5. $\phi = 4xz^3 - 3x^2 y^2 z$ *at* $(2, -1, 2)$ *along* $2i - 3j + 6k$

6. $\phi = \dfrac{xz}{x^2 + y^2}$ *at* $(1, -1, 1)$ *in the direction of* $\vec{A} = i - 2j + k$

7. $f(x, y, z) = x^2 y^2 z^2$ *at the point* $(1, 1, -1)$ *in the direction of the tangent to the curve* $x = e^t,\ y = 1 + 2\sin t,\ z = t - \cos t$ where $-1 \le t \le 1$

4. $\phi = x^2 yz + 4xz^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $$\nabla\phi = (2xyz + 4z^2)\, i + (x^2 z)\, j + (x^2 y + 8xz)\, k$$

$$[\nabla\phi]_{(1,\,-2,\,-1)} = 8i - j - 10k$$

The unit vector in the direction of $2i - j - 2k$ is

$$\hat{n} = \frac{2i - j - 2k}{\sqrt{4+1+4}} = \frac{2i - j - 2k}{3}$$

∴ the required directional derivative is

$$\nabla\phi \cdot \hat{n} = (8i - j - 10k) \cdot \frac{(2i - j - 2k)}{3}$$

Thus $$\nabla\phi \cdot \hat{n} = \frac{(8)(2) + (-1)(-1) + (-10)(-2)}{3} = \frac{37}{3}$$

5. $\phi = 4xz^3 - 3x^2 y^2 z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $$\nabla\phi = (4z^3 - 6xy^2 z)\, i - (6x^2 yz)\, j + (12xz^2 - 3x^2 y^2)\, k$$

$$[\nabla\phi]_{(2,\,-1,\,2)} = 8i + 48j + 84k$$

The unit vector in the direction of $2i - 3j + 6k$ is

$$\hat{n} = \frac{2i - 3j + 6k}{\sqrt{4+9+36}} = \frac{2i - 3j + 6k}{7}$$

the required directional derivative is

$$\nabla\phi \cdot \hat{n} = (8i + 48j + 84k) \cdot \frac{2i - 3j + 6k}{7}$$

Thus $$\nabla\phi \cdot \hat{n} = \frac{(8)(2) + (48)(-3) + (84)(6)}{7} = \frac{376}{7}$$

6. $\phi = \dfrac{xz}{x^2 + y^2}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $$\nabla\phi = z\left\{\frac{(x^2+y^2)\,1-x\,(2x)}{(x^2+y^2)^2}\right\}i+\left\{\frac{-2xyz}{(x^2+y^2)^2}\right\}j+\left(\frac{x}{x^2+y^2}\right)k$$

ie., $$\nabla\phi = \frac{z\,(y^2-x^2)}{(x^2+y^2)^2}\,i-\frac{2xyz}{(x^2+y^2)^2}\,j+\frac{x}{x^2+y^2}\,k$$

$$[\nabla\phi]_{(1,\,-1,\,1)} = \frac{1}{2}j+\frac{1}{2}k = \frac{1}{2}(j+k)$$

The unit vector normal in the direction of $\vec{A} = i-2j+k$ is

$$\hat{n} = \frac{i-2j+k}{\sqrt{1+4+1}} = \frac{i-2j+k}{\sqrt{6}}$$

Thus the required directional derivative is

$$\nabla\phi\cdot\hat{n} = \frac{1}{2}(j+k)\cdot\frac{(i-2j+k)}{\sqrt{6}} = \frac{0-2+1}{2\sqrt{6}} = \frac{-1}{2\sqrt{6}}$$

---

7. $f = x^2y^2z^2$

$$\therefore\quad \nabla f = \frac{\partial f}{\partial x}\,i+\frac{\partial f}{\partial y}\,j+\frac{\partial f}{\partial z}\,k$$

ie., $$\nabla f = 2xy^2z^2\,i+2x^2yz^2\,j+2x^2y^2z\,k$$

$$[\nabla f]_{(1,\,1,\,-1)} = 2i+2j-2k = 2\,(i+j-k) \qquad \ldots(1)$$

In order to find the direction of the tangent, let $\vec{r} = xi+yj+zk$

ie., $$\vec{r} = e^t\,i+(1+2\sin t)\,j+(t-\cos t)\,k$$

$$\frac{d\vec{r}}{dt} = e^t\,i+2\cos t\,j+(1+\sin t)\,k \text{ is the tangent vector}$$

We have $P = (x,\,y,\,z) = (1,\,1,\,-1)$ by data.

$$\therefore\quad e^t = 1\ ;\ 1+2\sin t = 1\ ;\ t-\cos t = -1$$

Here $e^t = 1 \Rightarrow t = 0$ and this value of t satisfy all the equations.

Thus $\left.\dfrac{d\vec{r}}{dt}\right|_{t=0} = i+2j+k$ is the direction of the tangent and the unit vector in this direction is

$$\hat{n} = \frac{i+2j+k}{\sqrt{1+4+1}} = \frac{i+2j+k}{\sqrt{6}}$$

$\therefore$ the required directional derivative of $f(x,\,y,\,z)$ along the tangent to the given curve is

The equation of the normal line is

$$\frac{x-1}{3} = \frac{y-1}{3} = \frac{z-1}{3} \quad \text{or} \quad x-1 = y-1 = z-1$$

---

9. *In which direction the directional derivative of $x^2 y z^3$ is maximum at $(2, 1, -1)$ and find the magnitude of this maximum*

>> We know that the directional derivative is maximum along the normal vector which being $\nabla\phi$.

Let $\quad \phi = x^2 y z^3$ so that we have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = 2xyz^3\, i + x^2 z^3\, j + 3x^2 yz^2\, k$$

$[\nabla\phi]_{(2,\, 1,\, -1)} = -4\,i - 4\,j + 12\,k$ which is the required direction in which the directional derivative is maximum. The magnitude of this is given by $\sqrt{4^2(1+1+9)} = 4\sqrt{11}$

---

10. [illegible] *direction that a person standing at the origin should move to get warm as quickly* [illegible] *le given that the temperature field is $x \sin z - y \cos z$*

>> Let $\phi = x \sin z - y \cos z$ be the temperature field.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie, $\quad \nabla\phi = \sin z\, i - \cos z\, j + (x \cos z + y \sin z)\, k$

$[\nabla\phi]_{(0,\, 0,\, 0)} = -j$ **is the required direction** which means to say that the person should move down along the y axis to get warm quickly

---

11. *If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(-1, 1, 2)$ has a maximum magnitude of 32 units in the direction parallel to y-axis find a, b, c*

>> Maximum directional derivative is along $\nabla\phi$ and in the direction parallel to y-axis the magnitude is given to be 32 units.

$\therefore \quad \nabla\phi \cdot j = 32$ at $(-1, 1, 2)$ $\quad \ldots(1)$

We have $\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$

ie., $\quad \nabla\phi = (ay^2 + 3cx^2z^2)\, i + (2axy + bz)\, j + (by + 2cx^3 z)\, k$

$$[\nabla\phi]_{(-1,\, 1,\, 2)} = (a + 12c)\, i + (-2a + 2b)\, j + (b - 4c)\, k$$

Now $\quad \nabla\phi \cdot j = -2a + 2b = 32$ by using (1), or $-a + b = 16$

Also since $\nabla\phi$ is parallel to the y - axis we must have
$a + 12c = 0$ and $b - 4c = 0$

Thus by solving the three equations :

$$-a + b = 16, \; a + 12c = 0, \; b - 4c = 0 \quad \text{we obtain}$$

$$a = -12, \quad b = 4, \quad c = 1$$

---

12. *Find the angle between the surfaces* [illegible] *at the point* $(2, -1, 2)$

>> The angle between the surfaces is defined to be equal to the angle between their normals and we know that $\nabla\phi$ is a vector normal to the surface. We have the equation of the two surfaces given by

$$x^2 + y^2 + z^2 = 9 \text{ and } x^2 + y^2 - z = 3$$

Let $\quad \phi_1 = x^2 + y^2 + z^2$ and $\phi_2 = x^2 + y^2 - z$

We have $\nabla\phi = \dfrac{\partial\phi}{\partial x}\, i + \dfrac{\partial\phi}{\partial y}\, j + \dfrac{\partial\phi}{\partial z}\, k$

$\therefore \quad \nabla\phi_1 = 2x\,i + 2y\,j + 2zk$ and $\nabla\phi_2 = 2x\,i + 2y\,j - k$

$\therefore \quad [\nabla\phi_1]_{(2,\,-1,\,2)} = 4i - 2j + 4k = 2(2i - j + 2k)$

$\quad [\nabla\phi_2]_{(2,\,-1,\,2)} = 4i - 2j - k$

If θ is the angle between these two normals we have

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\;|\nabla\phi_2|}$$

ie., $\quad \cos\theta = \dfrac{2(8 + 2 - 2)}{\sqrt{2^2(4 + 1 + 4)}\;\sqrt{16 + 4 + 1}} = \dfrac{8}{3\sqrt{21}}$

Thus $\quad \theta = \cos^{-1}(8/3\sqrt{21})$

---

13. *Find the angle* [illegible] $(\ldots, 1, 2)$ *and* $(3, 3, -3)$

>> Let $\phi = xy - z^2$ and we know that $\nabla\phi$ is a vector normal to the surface

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = y\,i + x\,j - 2z\,k$$

$\therefore \quad [\nabla\phi]_{(4,\,1,\,2)} = i + 4j - 4k = \vec{A}$ *(say)*

$$[\nabla\phi]_{(3,\,3,\,-3)} = 3i + 3j + 6k = 3\,(i + j + 2k) \cdot\cdot\, \vec{B} \quad (say)$$

If θ is the angle between the vectors $\overline{A}$ and $\overline{B}$ we have

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\;|\vec{B}|}$$

Now, $$\cos\theta = \frac{3\,(1 + 4 - 8)}{\sqrt{1+16+16}\;\sqrt{3^2\,(1+1+4)}}$$

$$= \frac{-3}{\sqrt{33}\;\sqrt{6}} = \frac{-3}{\sqrt{3}\;\sqrt{11}\;\sqrt{3}\;\sqrt{2}}$$

Thus $\cos\theta = \dfrac{-1}{\sqrt{22}}$ or $\theta = \pi \pm \cos^{-1}\left(\dfrac{1}{\sqrt{22}}\right)$

---

14. *Find the values of the constants a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2 y + z^3 = 4$ are orthogonal at the point $(1, -1, 2)$*

>> First we have to ensure that the given point lies on both the surfaces.

Substituting $(1, -1, 2)$ onto the equation $ax^2 - byz = (a+2)x$ we obtain

$$a + 2b = a + 2 \;\Rightarrow\; 2b = 2 \;\text{ or }\; b = 1$$

Also if $(1, -1, 2)$ is substituted onto the L.H.S of the equation $4x^2 y + z^3 = 4$ we get 4 which is equal to the R.H.S

$\Rightarrow$ the given point lies on both the surfaces when $b = 1$.

In order to find a we have to use the orthogonality condition $\nabla\phi_1 \cdot \nabla\phi_2 = 0$ where

$$\phi_1 = ax^2 - byz - (a+2)\,x \quad\text{and}\quad \phi_2 = 4x^2 y + z^3$$

Now $$\nabla\phi_1 = \{2ax - (a+2)\}\,i + (-bz)\,j + (-by)\,k$$

$$\nabla\phi_2 = 8xyi + 4x^2 j + 3z^2 k$$

$\therefore$ $$[\nabla\phi_1]_{(1,\,-1,\,2)} = (a-2)\,i - 2bj + bk$$

$$[\nabla\phi_2]_{(1,\,-1,\,2)} = -8i + 4j + 12k$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0 \text{ gives } -8\,(a-2) - 8b + 12b = 0$$

ie., $-8a + 4b + 16 = 0$. But $b = 1$ and hence we get $a = 5/2$

Thus $a = 5/2$ and $b = 1$ are the required values

---

15. If [illegible] $z^2 k$ and $\phi = 2z - x^3 y$, compute $\vec{A} \cdot \nabla\phi$, and $\vec{A} \times \nabla\phi$ at [illegible]

>> $$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k \quad ; \ \phi = 2z - x^3 y$$

$\therefore$ $$\nabla\phi = -3x^2 y\, i - x^3 j + 2k$$

$$[\nabla\phi]_{(1,-1,1)} = 3i - j + 2k$$

Now $$[\vec{A}]_{(1,-1,1)} = 2i + 3j + k$$

$\therefore$ $$[\vec{A} \cdot \nabla\phi]_{(1,-1,1)} = (3)(2) + (-1)(3) + (2)(1) = \mathbf{5}$$

Also $$\vec{A} \times \nabla\phi = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

ie., $$\vec{A} \times \nabla\phi = i(6+1) - j(4-3) + k(-2-9) = \mathbf{7i - j - 11k}$$

16. *Find the directional derivative of $\phi = x^2 y z^3$ at $(1, 1, 1)$ in*

(a) a direction which make equal angles with the co ordinate axes

(b) along the x-axis

>> [**Note** : *If a straight line make angles α, β, γ with the coordinate axes then* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *are the direction cosines of the line and it satisfy the identity* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ *The direction being* $\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$.]

(a) Consider $\phi = x^2 y z^3$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

ie., $$\nabla\phi = (2xyz^3)\, i + (x^2 z^3)\, j + (3x^2 y z^2)\, k$$

$$[\nabla\phi]_{(1,1,1)} = 2i + j + 3k$$

Since the directional line is making equal angles with the co ordinate axes we have

$$\alpha = \beta = \gamma \text{ and hence } \cos\alpha = \cos\beta = \cos\gamma = a \quad (say)$$

$\therefore$ the direction $\vec{d} = ai + aj + ak$ and the associated unit vector $\hat{n} = \vec{d} / |\vec{d}|$

ie., $$\hat{n} = \frac{a(i + j + k)}{\sqrt{a^2(1+1+1)}} = \frac{i + j + k}{\sqrt{3}}$$

Hence the required directional derivative is

$$\nabla\phi \cdot \hat{n} = (2i + j + 3k) \cdot \frac{i + j + k}{\sqrt{3}} = \frac{6}{\sqrt{3}} = \mathbf{2\sqrt{3}}$$

(b) Unit vector along the x - axis is $\hat{i}$ and hence the required directional derivative is $\nabla\phi \cdot \hat{i}$. Thus we get,

$$(2i + j + 3k) \cdot i = 2$$

17. [illegible] at the point (2, 1, 1)

>> The normal derivative at (2, 1, 1) = $|\nabla\phi|$ at (2, 1, 1)

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $\nabla\phi = \log z\, i - 2y\, j + (x/z)\, k$ and $[\nabla\phi]_{(2,\,1,\,1)} = -2j + 2k$

Thus $|\nabla\phi| = 2\sqrt{2}$

18. *If f and g are scalar point functions of x, y, z prove the following*

(a) $\nabla(fg) = f\,\nabla g + g\,\nabla f$

(b) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\,\nabla f - f\,\nabla g}{g^2}$ *where* $g \neq 0$

>> (a) $\nabla(fg) = \Sigma \dfrac{\partial}{\partial x}(fg)\, i = \Sigma\left(f\dfrac{\partial g}{\partial x} + g\dfrac{\partial f}{\partial x}\right) i$

$$= f\,\Sigma \frac{\partial g}{\partial x}\, i + g\,\Sigma \frac{\partial f}{\partial x}\, i$$

Thus $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(b) $$\nabla\left(\frac{f}{g}\right) = \Sigma \frac{\partial}{\partial x}\left(\frac{f}{g}\right) i = \Sigma \frac{g\dfrac{\partial f}{\partial x} - f\dfrac{\partial g}{\partial x}}{g^2}\, i$$

$$= \frac{1}{g^2}\left\{g\,\Sigma \frac{\partial f}{\partial x}\, i - f\,\Sigma \frac{\partial g}{\partial x}\, i\right\}$$

Thus $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\,\nabla f - f\,\nabla g}{g^2}$

19. *Find* div $\vec{F}$ *and* curl $\vec{F}$ *where* $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

>> Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\vec{F} = \text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

$\therefore \quad \vec{F} = (3x^2 - 3yz)\, i + (3y^2 - 3xz)\, j + (3z^2 - 3xy)\, k$

Now $\quad \text{div } \vec{F} = \nabla \cdot \vec{F}$

ie., $\quad = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \{ (3x^2 - 3yz)\, i + (3y^2 - 3xz)\, j + (3z^2 - 3xy)\, k$

$$= \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3xz) + \frac{\partial}{\partial z} (3z^2 - 3xy)$$

Thus $\quad \text{div } \vec{F} = 6x + 6y + 6z = 6(x + y + z)$

Also $\quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$

$$= i \left\{ \frac{\partial}{\partial y} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3y^2 - 3xz) \right\}$$

$$- j \left\{ \frac{\partial}{\partial x} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3x^2 - 3yz) \right\} + k \left\{ \frac{\partial}{\partial x} (3y^2 - 3xz) - \frac{\partial}{\partial y} (3x^2 - 3yz) \right\}$$

$$= i \{-3x - (-3x)\} - j \{-3y - (-3y)\} + k \{-3z - (-3z)\}$$

ie., $\quad \text{curl } \vec{F} = \vec{0}$

Thus **div** $\vec{F} = 6(x + y + z)$; **curl** $\vec{F} = \vec{0}$

---

20. *If* $\vec{F} = \nabla (x y^3 z^2)$ *find* div $\vec{F}$ *and* curl $\vec{F}$ *at the point* $(1, -1, 1)$.

>> Let $\phi = x y^3 z^2$

$$\vec{F} = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = y^3 z^2 i + 3xy^2 z^2 j + 2xy^3 z k$$

$\text{div } \vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \left(y^3 z^2 i + 3xy^2 z^2 j + 2xy^3 z k \right)$$

$$= \frac{\partial}{\partial x} (y^3 z^2) + \frac{\partial}{\partial y} (3xy^2 z^2) + \frac{\partial}{\partial z} (2xy^3 z)$$

$$= 0 + 6xyz^2 + 2xy^3 = 2xy(3z^2 + y^2)$$

$\therefore \quad \text{div } \vec{F} \text{ at } (1, -1, 1) = -2(3 + 1) = -8$

Also, $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^3 z^2 & 3xy^2 z^2 & 2xy^3 z \end{vmatrix}$

$$= i\,(6xy^2 z - 6xy^2 z) - j\,(2y^3 z - 2y^3 z) + k\,(3y^2 z^2 - 3y^2 z^2)$$

Thus **curl** $\vec{F} = \vec{0}$

21. *If* $\vec{F} = (3x^2 y - z)\,i + (xz^3 + y^4)\,j - 2x^3 z^2 k$, *find grad* $(\text{div } \vec{F})$ *at* $(2, -1, 0)$

>> $\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\dfrac{\partial}{\partial x} i + \dfrac{\partial}{\partial y} j + \dfrac{\partial}{\partial z} k \right)$

$$\cdot \left\{ (3x^2 y - z)\, i + (xz^3 + y^4)\, j - 2x^3 z^2 k \right\}$$

$$= \frac{\partial}{\partial x}(3x^2 y - z) + \frac{\partial}{\partial y}(xz^3 + y^4) + \frac{\partial}{\partial z}(-2x^3 z^2)$$

$$\nabla \cdot \vec{F} = 6xy + 4y^3 - 4x^3 z = \phi \ (say)$$

Now grad $(\text{div } \vec{F}) = \text{grad } \phi = \nabla\phi$

We have $\nabla\phi = \dfrac{\partial\phi}{\partial x} i + \dfrac{\partial\phi}{\partial y} j + \dfrac{\partial\phi}{\partial z} k$

ie., $\nabla\phi = (6y - 12x^2 z)\, i + (6x + 12y^2)\, j + (-4x^3)\, k$

Thus $[\nabla\phi]_{(2,\,-1,\,0)} = \mathbf{-6i + 24j - 32k}$

22. *If* $\vec{A} = xz^3 i - 2x^2 yz\, j + 2yz^4 k$ *find* $\nabla \cdot \vec{A}, \nabla \times \vec{A}$ *and* $\nabla \cdot (\nabla \times \vec{A})$ *at* $(1, -1, 1)$

>> $\nabla \cdot \vec{A} = \left(\dfrac{\partial}{\partial x} i + \dfrac{\partial}{\partial y} j + \dfrac{\partial}{\partial z} k \right) \cdot \left(xz^3 i - 2x^2 yz\, j + 2yz^4 k \right)$

$$= \frac{\partial}{\partial x}(xz^3) - \frac{\partial}{\partial y}(2x^2 yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$= z^3 - 2x^2 z + 8yz^3$$

$\therefore$ $\nabla \cdot \vec{A}$ or div $\vec{A}$ at $(1, -1, 1) = 1 - 2 - 8 = -9$

Next $\quad \nabla \times \vec{A}$ or curl $\vec{A} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$

$$= i(2z^4 + 2x^2y) - j(0 - 3xz^2) + k(-4xyz - 0)$$

$$= 2(x^2y + z^4)\, i + 3xz^2 j - 4xyz\, k$$

$\therefore \quad \nabla \times \vec{A}$ at $(1, -1, 1) = 3j + 4k$

Now $\quad \nabla \cdot (\nabla \times \vec{A})$ or div (curl $\vec{A}$)

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \{2(x^2y + z^4)\, i + 3xz^2 j - 4xyz\, k\}$$

$$= \frac{\partial}{\partial x}\{2(x^2y + z^4)\} + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-4xyz)$$

$$= 4xy - 4xy = 0$$

Thus $\quad \nabla \cdot \vec{A} = -9$ and $\nabla \cdot (\nabla \times \vec{A}) = 0$

23. *If $\vec{F} = (x + y + 1)\, i + j - (x + y)\, k$ show that $\vec{F} \cdot$ curl $\vec{F} = 0$*

$\gg \quad$ curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x + y + 1) & 1 & -x - y \end{vmatrix}$

$$= i(-1 - 0) - j(-1 - 0) + k(0 - 1)$$

ie., $\quad$ curl $\vec{F} = -i + j - k$

$\therefore \quad \vec{F} \cdot$ curl $\vec{F} = \{(x + y + 1)\, i + j - (x + y)\, k\} \cdot (-i + j - k)$

ie.,
$$= (x + y + 1)(-1) + (1)(1) + (x + y)(1)$$

$$= -x - y - 1 + 1 + x + y = 0$$

Thus $\quad \vec{F} \cdot$ curl $\vec{F} = 0 \Rightarrow \vec{F}$ is perpendicular to curl $\vec{F}$

24. *Find* curl (curl $\vec{A}$) *given that* $\vec{A} = xy\,i + y^2 z\,j + z^2 y\,k$

>> $$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & y^2 z & z^2 y \end{vmatrix}$$

$$= i\,(z^2 - y^2) - j\,(0 - 0) + k\,(0 - x)$$

$$\text{curl } \vec{A} = (z^2 - y^2)\,i - x\,k$$

Now $\text{curl } (\text{curl } \vec{A}) = \nabla \times (\nabla \times \vec{A})$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (z^2 - y^2) & 0 & -x \end{vmatrix} = i\,(0 - 0) - j\,(-1 - 2z) + k\,(0 + 2y)$$

$$\therefore \quad \textbf{curl (curl } \vec{A}\textbf{)} = \mathbf{(1 + 2z)\,j + 2y\,k}$$

---

25. *If* $\vec{V} = \dfrac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$ *then show that* div $\vec{V} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ *and* curl $\vec{V} = \vec{0}$

>> Observing the symmetric nature in $\vec{V}$, we can write

$$\text{div } \vec{V} = \nabla \cdot V = \left(\Sigma \frac{\partial}{\partial x}\, i \right) \cdot \left(\Sigma \frac{xi}{\sqrt{x^2 + y^2 + z^2}} \right)$$

ie., $$= \Sigma \frac{\partial}{\partial x} \left[\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right]$$

$$= \Sigma \frac{\sqrt{x^2 + y^2 + z^2} - (x)\, \dfrac{1}{2\sqrt{x^2 + y^2 + z^2}}\,(2x)}{x^2 + y^2 + z^2}$$

$$= \Sigma \frac{(x^2 + y^2 + z^2) - x^2}{(x^2 + y^2 + z^2)^{3/2}} = \Sigma \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

ie., $$= \frac{(y^2 + z^2) + (z^2 + x^2) + (x^2 + y^2)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2\,(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore \quad \text{div } \vec{V} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

(**Note** · *We should not remove the common factor appearing in the third row of the determinant as differential operators are involved in the second row*)

$$\therefore \quad \text{curl } \vec{V} = \Sigma\, i \left\{ \frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) \right\}$$

$$= \Sigma\, i \left\{ z\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-3/2}\,(2y) - y\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-3/2}\,(2z) \right\}$$

$$= \Sigma\, i \left\{ -yz(x^2+y^2+z^2)^{-3/2} + yz(x^2+y^2+z^2)^{-3/2} \right\} = \vec{0}$$

Thus $\quad$ **curl** $\vec{V} = \vec{0}$

26. *If* $\vec{V} = \vec{\omega} \times \vec{r}$ *prove that* curl $\vec{V} = 2\vec{\omega}$ *where* $\vec{\omega}$ *is a constant vector.*

>> Let $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$ be the constant vector.

We have $\quad \vec{r} = x\,i + y\,j + z\,k$

$$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \Sigma\, i\,(\omega_2 z - \omega_3 y)$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= \Sigma\, i\,\Big(\omega_1 - (-\omega_1)\Big) = \Sigma\, 2\omega_1 i = 2(\omega_1 i + \omega_2 j + \omega_3 k) = \mathbf{2}\vec{\omega}$$

$\therefore \qquad \textbf{curl } \vec{V} = 2\,\vec{\omega} \quad \text{or} \quad \vec{\omega} = \frac{1}{2}\, \textbf{curl } \vec{V}$

Remark : *We had referred to this example while giving the physical meaning of "curl". The theoretical version of this problem is as follows :*

When a rigid body is in motion the angular velocity is equal to half the curl of its linear velocity at any point.

If $\vec{A}$ is a constant vector prove the following.

27. $\text{grad}\ (\vec{A} \cdot \vec{r}) = \vec{A}$

28. $\text{div}\ (\vec{A} \times \vec{r}) = 0$

29. $\text{curl}\ (\vec{A} \times \vec{r}) = 2\,\vec{A}$

30. $\text{curl}\ \left((\vec{A} \cdot \vec{r})\, \vec{r} \right) = \vec{A} \times \vec{r}$

31. $\text{div}\ \vec{A} \times (\vec{r} \times \vec{A}) = 2\,|\vec{A}$

27. Let $\vec{A} = a_1 i + a_2 j + a_3 k \quad ; \quad \vec{r} = x i + y j + z k$

(**Note** : *This step is common for all the five examples*)

$$\vec{A} \cdot \vec{r} = \Sigma\, a_1 x$$

$$\therefore \qquad \nabla\, (\vec{A} \cdot \vec{r}) = \left(\Sigma\, \frac{\partial}{\partial x}\, i \right) (\Sigma\, a_1 x) = \Sigma\, a_1 i = \vec{A}$$

Thus $\quad \nabla\, (\vec{A} \cdot \vec{r}) = \vec{A} \quad$ or $\quad \textbf{grad}\ (\vec{A} \cdot \vec{r}) = \vec{A}$

$$\textbf{28.}\ \vec{A} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \Sigma\, i\,(a_2 z - a_3 y)$$

$$\therefore \qquad \nabla \cdot (\vec{A} \times \vec{r}) = \left(\Sigma\, \frac{\partial}{\partial x}\, i \right) \cdot \Sigma\, i\,(a_2 z - a_3 y)$$

$$= \Sigma\, \frac{\partial}{\partial x}\,(a_2 z - a_3 y) = 0 + 0 + 0 = 0$$

Thus $\quad \nabla \cdot (\vec{A} \times \vec{r}) = 0 \quad$ or $\quad \textbf{div}\ (\vec{A} \times \vec{r}) = 0$

---

29. This example is same as Example-26 as we have $\vec{A}$ in place of $\vec{\omega}$ being a constant vector. We have proved in Example-26 that curl $(\vec{\omega} \times \vec{r}) = 2\,\vec{\omega}$

Thus $\quad \textbf{curl}\ (\vec{A} \times \vec{r}) = 2\,\vec{A} \quad$ or $\quad \nabla \times (\vec{A} \times \vec{r}) = 2\,\vec{A}$

---

30. $\vec{A}\cdot\vec{r} = a_1 x + a_2 y + a_3 z$

Also $\quad (\vec{A}\cdot\vec{r})\,\vec{r} = (a_1 x + a_2 y + a_3 z)\,(xi + yj + zk)$

ie., $\quad (\vec{A}\cdot\vec{r})\,\vec{r} = \Sigma\,(a_1 x^2 + a_2 xy + a_3 xz)\,i$

Now $\quad \text{curl}\,\{(\vec{A}\cdot\vec{r})\,\vec{r}\} = \nabla \times \{(\vec{A}\cdot\vec{r})\,\vec{r}\}$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (a_1 x^2 + a_2 xy + a_3 xz) & (a_2 y^2 + a_3 yz + a_1 xy) & (a_3 z^2 + a_1 zx + a_2 yz) \end{vmatrix}$$

$$= \Sigma i\,(a_2 z - a_3 y) \qquad \ldots (1)$$

Now consider R.H.S $= \vec{A}\times\vec{r}$

ie., $\quad = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \Sigma\,(a_2 z - a_3 y) \qquad \ldots (2)$

Comparing (1) and (2) we have,

$$\text{curl}\,\{(\vec{A}\cdot\vec{r})\,\vec{r}\} = \vec{A}\times\vec{r} \quad \text{or} \quad \nabla\times\{(\vec{A}\cdot\vec{r})\,\vec{r}\} = \vec{A}\times\vec{r}$$

31. $\vec{r}\times\vec{A} = \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i\,(a_3 y - a_2 z)$

$$\vec{A}\times(\vec{r}\times\vec{A}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ (a_3 y - a_2 z) & (a_1 z - a_3 x) & (a_2 x - a_1 y) \end{vmatrix}$$

$$= \Sigma\, i\,(a_2^2 x - a_1 a_2 y - a_1 a_3 z + a_3^2 x)$$

Now $\quad \nabla\cdot\{\vec{A}\times(\vec{r}\times\vec{A})\}$

$$= \left(\Sigma\,\frac{\partial}{\partial x}\,i\right)\cdot\Sigma\, i\left(a_2^2 x - a_1 a_2 y - a_1 a_3 z + a_3^2 x\right)$$

$$= \Sigma\,\frac{\partial}{\partial x}\,(a_2^2 x - a_1 a_2 y - a_1 a_3 z + a_3^2 x)$$

$$= \Sigma\,(a_2^2 + a_3^2) = (a_2^2 + a_3^2) + (a_3^2 + a_1^2) + (a_1^2 + a_2^2)$$

$$= 2\,(a_1^2 + a_2^2 + a_3^2) = 2\,|\vec{A}\,|^2$$

$$\because \quad |\vec{A}\,| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad \Rightarrow \quad |\vec{A}\,|^2 = a_1^2 + a_2^2 + a_3^2$$

Thus $\quad \mathbf{div}\,\{\vec{A} \times (\vec{r} \times \vec{A}\,)\} = 2\,|\vec{A}\,|^2$

Remark : *In all these five examples we must observe the symmetry and use "sigma" notation to arrive at the desired result quickly.*

If $\quad \vec{r} = xi + yj + zk \quad$ *and* $\; r = |\vec{r}\,| \;$ *prove the following*

32. $\nabla\,(r^n) = n\,r^{n-2}\,\vec{r}$

33. $\nabla \cdot (r^n\,\vec{r}\,) = (n+3)\,r^n$

34. $\nabla \times (r^n\,\vec{r}\,) = 0$

35. $\nabla^2\,(r^n) = n\,(n+1)\,r^n$

Note : *These are also a set of problems based on the aspect of symmetry and the problems are worked in the general form involving "n". Particular cases for n can also be asked. The notation and meaning concerning $\vec{r}$ and r are standard which need not be explicitly mentioned while asking these problems or particular cases of these.*

32. $r = |\vec{r}\,| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow \quad r^2 = x^2 + y^2 + z^2$ and differentiating partially w.r.t. x we get

$$2r\,\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{Also} \quad \frac{\partial r}{\partial y} = \frac{y}{r}\,, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Note : *The expression for these three partial derivatives will be of use in all the problems and the same will not be worked in every problem.*

Now $\quad \nabla\,(r^n) = \left(\Sigma\,\frac{\partial}{\partial x}\,i\right)(r^n)$

$$= \Sigma\,n\,r^{n-1}\,\frac{\partial r}{\partial x}\,i = \Sigma\,n\,r^{n-1}\left(\frac{x}{r}\right) i$$

$$= \Sigma\,n\,r^{n-2}\,x\,i = n\,r^{n-2}\,\Sigma\,x\,i = n\,r^{n-2}\,\vec{r}$$

Thus $\quad \nabla\,(r^n) = \mathbf{grad}\,(r^n) = n\,r^{n-2}\,\vec{r}$

33. $r^n \vec{r} = r^n \Sigma\, xi = \Sigma\, (r^n x)\, i$

$\therefore \qquad \nabla \cdot (r^n \vec{r}) = \left(\Sigma \dfrac{\partial}{\partial x}\, i\right) \cdot \Sigma\, (r^n x)\, i$

$$= \Sigma \frac{\partial}{\partial x}(r^n x) = \Sigma \left(r^n + n\, r^{n-1} \frac{\partial r}{\partial x}\, x \right)$$

ie., $\qquad = \Sigma \left(r^n + n\, r^{n-1} \dfrac{x}{r}\, x \right) = \Sigma \left(r^n + n\, r^{n-2} x^2 \right)$

On expanding the summation we get,

$$(r^n + n\, r^{n-2} x^2) + (r^n + n\, r^{n-2} y^2) + (r^n + n r^{n-2} z^2)$$

$$= 3\, r^n + n\, r^{n-2} (x^2 + y^2 + z^2)$$

$$= 3r^n + n\, r^{n-2}\, r^2 = 3\, r^n + n\, r^n = (n+3)\, r^n$$

Thus $\quad \boldsymbol{\nabla \cdot (r^n \vec{r}) = \textbf{div}\, (r^n \vec{r}) = (n+3)\, r^n}$

---

34. $r^n \vec{r} = r^n \Sigma\, x\, i = \Sigma\, (r^n x)\, i$

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \Sigma\, i \left\{ \frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y) \right\}$$

$$= \Sigma\, i \left\{ n\, r^{n-1} \frac{\partial r}{\partial y}\, z - n\, r^{n-1} \frac{\partial r}{\partial z}\, y \right\}$$

$$= \Sigma\, i \left\{ n\, r^{n-1} \frac{y}{r}\, z - n\, r^{n-1} \frac{z}{r}\, y \right\}$$

$$= \Sigma\, i\, (n\, r^{n-2} yz - n\, r^{n-2} yz) = \vec{0}$$

Thus $\quad \boldsymbol{\nabla \times (r^n \vec{r}) = \textbf{curl}\, (r^n \vec{r}) = \vec{0}}$

---

35. $\nabla^2 (r^n) = \Sigma \dfrac{\partial^2}{\partial x^2} (r^n) = \Sigma \dfrac{\partial}{\partial x} \dfrac{\partial}{\partial x} (r^n)$

$$= \Sigma \frac{\partial}{\partial x} \left(n\, r^{n-1} \frac{\partial r}{\partial x} \right) = \Sigma \frac{\partial}{\partial x} \left\{ n\, r^{n-1} \left(\frac{x}{r} \right) \right\}$$

ie., $$= \Sigma \frac{\partial}{\partial x} \left(n\, r^{n-2} x \right) = n\, \Sigma \left(r^{n-2} + (n-2)\, r^{n-3} \frac{\partial r}{\partial x} x \right)$$

ie., $$= n\, \Sigma \left(r^{n-2} + (n-2)\, r^{n-3} \frac{x}{r} x \right) = n\, \Sigma \left(r^{n-2} + (n-2)\, r^{n-4} x^2 \right)$$

On expanding the summation we get,

$$n \left\{ \left(r^{n-2} + (n-2)\, r^{n-4} x^2 \right) + \left(r^{n-2} + (n-2)\, r^{n-4} y^2 \right) + \left(r^{n-2} + (n-2)\, r^{n-4} z^2 \right) \right\}$$

$$= n \left\{ 3\, r^{n-2} + (n-2)\, r^{n-4} (x^2 + y^2 + z^2) \right\}$$

$$= n \left\{ 3\, r^{n-2} + (n-2)\, r^{n-4}\, r^2 \right\}$$

$$= n \left\{ 3\, r^{n-2} + (n-2)\, r^{n-2} \right\} = n r^{n-2} (3 + n - 2)$$

$$= n\, r^{n-2} (n+1)$$

Thus $\boldsymbol{\nabla^2 (r^n) = n (n+1)\, r^{n-2}}$

Note : *A few important particular cases of these are given and is left as an exercise for the reader to work out these independently.*

Prove with usual meanings the following :

1. $\nabla \left(\dfrac{1}{r} \right) = \dfrac{-\vec{r}}{r^3}$ 2. $\nabla \cdot \left(\dfrac{\vec{r}}{r^3} \right) = 0$ 3. $\nabla^2 \left(\dfrac{1}{r} \right) = 0$

---

36. *Show that* $\nabla^2 [f(r)] = f''(r) + \dfrac{2}{r} f'(r)$ *and hence deduce the expressions for* $\nabla^2 (e^r)$ *and* $\nabla^4 (e^r)$.

>> $$\nabla^2 [f(r)] = \Sigma \frac{\partial^2}{\partial x^2} [f(r)] = \Sigma \frac{\partial}{\partial x} \frac{\partial}{\partial x} [f(r)]$$

ie., $$= \Sigma \frac{\partial}{\partial x} \left\{ f'(r) \frac{\partial r}{\partial x} \right\} = \Sigma \frac{\partial}{\partial x} \left\{ f'(r) \frac{x}{r} \right\}$$

$$\therefore \qquad r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\nabla^2[f(r)] = \Sigma\left\{f'(r)\left[\frac{r - x\dfrac{\partial r}{\partial x}}{r^2}\right] + f''(r)\frac{\partial r}{\partial x}\,\frac{x}{r}\right\}$$

ie.,
$$= \Sigma f'(r)\left[\frac{r - x\left(\dfrac{x}{r}\right)}{r^2}\right] + \Sigma f''(r)\,\frac{x}{r}\,\frac{x}{r}$$

$$= \Sigma\frac{f'(r)}{r^3}(r^2-x^2) + \Sigma f''(r)\frac{x^2}{r^2}$$

$$= \frac{f'(r)}{r^3}\{(r^2-x^2)+(r^2-y^2)+(r^2-z^2)\} + \frac{f''(r)}{r^2}\Sigma x^2$$

$$= \frac{f'(r)}{r^3}\{3r^2-(x^2+y^2+z^2)\} + \frac{f''(r)}{r^2}r^2$$

$$= \frac{f'(r)}{r^3}(3r^2-r^2)+f''(r) = \frac{2}{r}f'(r)+f''(r)$$

Thus $\quad \nabla^2[f(r)] = \dfrac{2}{r}f'(r)+f''(r)$ $\qquad\qquad$ ...(1)

Now let $f(r) = e^r \quad \therefore \; f'(r) = e^r = f''(r)$

Thus (1) becomes

$$\nabla^2(e^r) = \frac{2}{r}e^r + e^r = e^r\left(\frac{2}{r}+1\right)$$

Thus $\quad \nabla^2(e^r) = e^r\left(\dfrac{2}{r}+1\right)$ $\qquad\qquad$ ...(2)

Also $\quad \nabla^4(e^r) = \nabla^2\,\nabla^2(e^r) = \nabla^2\left\{e^r\left(\dfrac{2}{r}+1\right)\right\}$

Now taking $f(r) = e^r\left(\dfrac{2}{r}+1\right)$

$$f'(r) = e^r\left(\frac{-2}{r^2}\right) + e^r\left(\frac{2}{r}+1\right) \qquad \text{...(3)}$$

$$f''(r) = e^r\left(\frac{4}{r^3}\right) + e^r\left(\frac{-2}{r^2}\right) + e^r\left(\frac{-2}{r^2}\right) + e^r\left(\frac{2}{r}+1\right)$$

ie., $$f''(r) = \frac{4}{r^3}e^r - \frac{4}{r^2}e^r + \frac{2}{r}e^r + e^r \qquad \ldots (4)$$

Substituting (3) and (4) in (1) we get

$$\nabla^2\left\{e^r\left(\frac{2}{r}+1\right)\right\}$$

$$= \frac{2}{r}\left\{-\frac{2e^r}{r^2} + \frac{2e^r}{r} + e^r\right\} + \frac{4}{r^3}e^r - \frac{4}{r^2}e^r + \frac{2}{r}e^r + e^r$$

Simplifying R.H.S we obtain

$$\nabla^2\{\nabla^2(e^r)\} = e^r\left(\frac{4}{r}+1\right)$$

Thus $$\nabla^4(e^r) = e^r\left(\frac{4}{r}+1\right)$$

37. [illegible]

>> We need to first obtain $\nabla \cdot \left(\frac{\vec{r}}{r^2}\right)$ similar to Example - 33.

On doing so we can obtain

$$\nabla \cdot \left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2} \qquad \textit{(particular case where } n = -2)$$

Then $\nabla^2\left(\frac{1}{r^2}\right)$ is to be worked out similar to Example-35

On doing so we can obtain, *(particular case where* $n = -2$) $\nabla^2\left(\frac{1}{r^2}\right) = \frac{2}{r^4}$

Thus $$\nabla^2\left\{\nabla \cdot \left(\frac{\vec{r}}{r^2}\right)\right\} = \frac{2}{r^4}$$

38. *If* A = [illegible]

>> We shall denote $r = |\vec{r}|$ so that $\vec{A} = \frac{\vec{r}}{r}$

$$\text{div } \vec{A} = \nabla \cdot \left(\frac{\vec{r}}{r}\right) \text{ is similar to Example - 33.}$$

On doing so we can obtain

$$\nabla \cdot \left(\frac{\vec{r}}{r}\right) = \frac{2}{r} \quad \text{(particular case where } n = -1)$$

$\therefore$ $$\text{grad } (\text{div } A) = \text{grad}\left(\frac{2}{r}\right) = 2\,\nabla\left(\frac{1}{r}\right)$$

$\nabla\left(\frac{1}{r}\right)$ is a particular case of Example - 32 for $n = -1$

We can obtain $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

Thus **grad (div $\vec{A}$) = $\nabla(\nabla \cdot \vec{A}) = -\frac{2\vec{r}}{r^3}$**

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39. *If $\hat{r}$ represents a unit vector in the direction of $\vec{r}$ prove that*

(i) $\nabla \cdot \hat{r} = \frac{2}{r}$ *(ii)* $\nabla \times \hat{r} = \vec{0}$

>> $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$

(i) $\nabla \cdot \hat{r} = \nabla \cdot \left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$ *(Refer Example-38)*

(ii) $\nabla \times \hat{r} = \nabla \times \left(\frac{\vec{r}}{r}\right)$ is a particular case of Example - 34 for $n = -1$

We can obtain $\nabla \times \left(\frac{\vec{r}}{r}\right) = 0$ or $\nabla \times \hat{r} = \vec{0}$

Thus **$\nabla \cdot \hat{r} = 2/r$ and $\nabla \times \hat{r} = \vec{0}$**

Remark : *Infact we have worked this example without using the notation $\vec{r}$ and $\vec{r}$. The same results have been obtained in Example - 25 where $\vec{V}$ is nothing but $\hat{r}$.*

---

40. *If* $\phi = 2x^3 y^2 z^4$ *find* div (grad ϕ) *and verify that* $\nabla \cdot (\nabla\phi) = \nabla^2 \phi$

>> Consider $\phi = 2x^3 y^2 z^4$

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $$\nabla\phi = (6x^2 y^2 z^4)\, i + (4x^3 y z^4)\, j + (8x^3 y^2 z^3)\, k$$

$$\text{div } (\text{grad } \phi) = \nabla \cdot \nabla\phi$$

ie., $$= \left(\frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k \right) \cdot \left(6x^2 y^2 z^4\, i + 4x^3 yz^4\, j + 8x^3 y^2 z^3\, k \right)$$

$$= \frac{\partial}{\partial x} (6x^2 y^2 z^4) + \frac{\partial}{\partial y} (4x^3 yz^4) + \frac{\partial}{\partial z} (8x^3 y^2 z^3)$$

$$\therefore \quad \nabla \cdot (\nabla \phi) = 12\, xy^2 z^4 + 4\, x^3 z^4 + 24\, x^3 y^2 z^2 \qquad \ldots (1)$$

Next $$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad \ldots (2)$$

Consider $\phi = 2x^3 y^2 z^4$

$$\frac{\partial \phi}{\partial x} = 6x^2 y^2 z^4, \quad \frac{\partial \phi}{\partial y} = 4x^3 yz^4, \quad \frac{\partial \phi}{\partial z} = 8x^3 y^2 z^3$$

$$\frac{\partial^2 \phi}{\partial x^2} = 12xy^2 z^4, \quad \frac{\partial^2 \phi}{\partial y^2} = 4x^3 z^4, \quad \frac{\partial^2 \phi}{\partial z^2} = 24x^3 y^2 z^2$$

Adding these results we have according to (2)

$$\nabla^2 \phi = 12xy^2 z^4 + 4x^3 z^4 + 24x^3 y^2 z^2 \qquad \ldots (3)$$

Thus by comparing (1) and (3) $\nabla \cdot \nabla\phi = \nabla^2 \phi$ **is verified.**

4.24 Solenoidal and Irrotational vector fields

We have already referred to these while giving the physical meaning of divergence and curl.

A vector field $\vec{F}$ is said to be **solenoidal if** div $\vec{F} = 0$ and **irrotational if** curl $\vec{F} = \vec{0}$.

Irrotational field is also called as *conservative field* or *potential field.*

When $\vec{F}$ is irrotational there always exists a scalar point function ϕ such that $\nabla\phi = \vec{F}$. Then ϕ is called a **scalar potential** of $\vec{F}$

WORKED PROBLEMS

41. *Show that* $\vec{F} = \dfrac{xi + yj}{x^2 + y^2}$ *is both solenoidal and irrotational*

>> div $\vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k \right) \cdot \left(\frac{x}{x^2 + y^2}\, i + \frac{y}{x^2 + y^2}\, j \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right)$$

$$= \left\{ \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} \right\} + \left\{ \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} \right\}$$

$$\frac{1}{(x^2+y^2)^2} \{ y^2 - x^2 + x^2 - y^2 \} = 0$$

Thus $\quad$ div $\vec{F} = 0 \Rightarrow \vec{F}$ **is solenoidal.**

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= 0i + 0j + k \left\{ \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2} \right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2} \right) \right\}$$

$$= k \left(\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} \right) = \vec{0}$$

Thus $\quad$ curl $\vec{F} = \vec{0} \Rightarrow \vec{F}$ **is irrotational.**

12. *Find the condition that the vector field $\vec{F} = (xyz)^b\,(x^a i + y^a j + z^a k)$ is both solenoidal and irrotational*

\>> $\quad \vec{F} = x^{a+b} y^b z^b\, i + x^b y^{a+b} z^b\, j + x^b y^b z^{a+b}\, k$

ie., $\quad \vec{F} = f_1 i + f_2 j + f_3 k$ *(say)* where

$$f_1 = x^{a+b} y^b z^b,\; f_2 = x^b y^{a+b} z^b,\; f_3 = x^b y^b z^{a+b} \qquad \ldots (1)$$

Now $\quad$ div $\vec{F} = 0 \Rightarrow \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z} = 0$

ie., $\quad (a+b)\, x^{a+b-1} y^b z^b + (a+b)\, x^b y^{a+b-1} z^b + (a+b)\, x^b y^b z^{a+b-1} = 0$

or $\quad (a+b)\,(xyz)^b \left\{ x^{a-1} + y^{a-1} + z^{a-1} \right\} = 0$

This equation is identically satisfied only when $a+b = 0$

Thus $\vec{F}$ **is solenoidal if $a+b=0$**

Next consider curl $\vec{F} = \vec{0}$

ie., $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \vec{0}$$

ie., $\Sigma \left(\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z} \right) i = \vec{0}$ and by using (1) we have

$$\Sigma \left(b x^b y^{b-1} z^{a+b} - b x^b y^{a+b} z^{b-1} \right) i = \vec{0}$$

ie., $$\Sigma\, b x^b \left(y^{b-1} z^{a+b} - y^{a+b} z^{b-1} \right) i = \vec{0}$$

Since $x^b \neq 0$, the equation is identically satisfied when $b = 0$ or $a + b = b - 1$

Thus $b = 0$ **or** $a = -1$ is the required condition for $\vec{F}$ **to be irrotational.**

If $\vec{F}$ is both solenoidal and irrotational we must have

$a + b = 0$ and $b = 0$ or $a + b = 0$ and $a = -1$.

ie., $a = 0$ **and** $b = 0$ **or** $a = -1$ **and** $b = 1$

Remark : *If* $a = 0$ *and* $b = 0$ *we have* $\vec{F} = i + j + k$

If $a = -1$, $b = 1$ *we have* $\vec{F} = yz\, i + zx\, j + xy\, k$

It can be easily seen that $div\, \vec{F} = 0$ *and* $curl\, \vec{F} = \vec{0}$ *in both the forms of* $\vec{F}$

function ϕ such that $F = \nabla \phi$

>> We have to show that curl $\vec{F} = \vec{0}$

$$\therefore \quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$= i(1-1) - j(1-1) + k(1-1) = \vec{0}$$

Hence $\vec{F}$ **is irrotational.**

Now let us consider $\nabla \phi = \vec{F}$

ie., $$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (y+z)\, i + (z+x)\, j + (x+y)\, k$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = y + z \qquad \therefore \quad \phi = \int (y+z)\, dx + f_1(y, z)$$

ie., $\phi = xy + xz + f_1(y, z)$ ... (1)

$$\frac{\partial \phi}{\partial y} = z + x \qquad \therefore \quad \phi = \int (z+x)\, dy + f_2(x, z)$$

ie., $\phi = yz + xy + f_2(x, z)$ ... (2)

$$\frac{\partial \phi}{\partial z} = x + y \qquad \therefore \quad \phi = \int (x+y)\, dz + f_3(x, y)$$

ie., $\phi = xz + yz + f_3(x, y)$ ... (3)

[*Now we need to suitably choose the arbitrary functions* $f_1(y, z)$, $f_2(x, z)$, $f_3(x, y)$ *such*

Similarly we have to choose $f_2(x, z)$ *from (1) and (3),* $f_3(x, y)$ *from (1) and (2).*]

Let us choose $f_1(y, z) = yz$, $f_2(z, x) = xz$, $f_3(x, y) = xy$ from (1), (2) & (3).

Thus the required $\phi = xy + yz + zx$

---

44. *Show that* $\vec{F} = (2xy^2 + yz)\, i + (2x^2y + xz + 2yz^2)\, j + (2y^2z + xy)\, k$ *is a conservative force field. Find its scalar potential.*

>> We have to show that curl $\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy^2 + yz) & (2x^2y + xz + 2yz^2) & (2y^2z + xy) \end{vmatrix}$$

$$= i(4yz + x - x - 4yz) - j(y - y) + k(4xy + z - 4xy - z) = \vec{0}$$

$\therefore$ $\vec{F}$ **is conservative.**

Now we have to find ϕ such $\nabla\phi = \vec{F}$

ie., $\dfrac{\partial \phi}{\partial x}\, i + \dfrac{\partial \phi}{\partial y}\, j + \dfrac{\partial \phi}{\partial z}\, k = (2xy^2 + yz)\, i + (2x^2y + xz + 2yz^2)\, j + (2y^2z + xy)\, k$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = 2xy^2 + yz \qquad \therefore \quad \phi = \int (2xy^2 + yz)\, dx + f_1(y, z)$$

ie., $\phi = x^2y^2 + xyz + f_1(y, z)$ ... (1)

$$\frac{\partial\phi}{\partial y} = 2x^2 y + xz + 2yz^2$$

$$\phi = \int (2x^2 y + xz + 2yz^2)\, dy + f_2(x, z)$$

ie., $\phi = x^2 y^2 + xyz + y^2 z^2 + f_2(x, z)$ ...(2)

$$\frac{\partial\phi}{\partial z} = 2y^2 z + xy \qquad \therefore\ \phi = \int (2y^2 z + xy)\, dz + f_3(x, y)$$

ie., $\phi = y^2 z^2 + xyz + f_3(x, y)$ ...(3)

Let us choose $f_1(y, z) = y^2 z^2$, $f_2(x, z) = 0$, $f_3(x, y) = x^2 y^2$ from (1), (2) & (3)

Thus $\phi = x^2 y^2 + y^2 z^2 + xyz$ is the required scalar potential

15. *Find the value of the constant 'a' such that the vector field.*
$(axy - z^3)\, i + (a - 2)\, x^2\, j + (1 - a)\, xz^2\, k$ *is irrotational and hence find a scalar function ϕ such that $\vec{F} = \nabla\phi$*

>> We have to find '*a*' such that curl $\vec{F} = \vec{0}$

ie.,
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (axy - z^3) & (a-2)\, x^2 & (1-a)\, xz^2 \end{vmatrix} = \vec{0}$$

ie., $i(0 - 0) - j\{(1 - a)\, z^2 + 3z^2\} + k\{(a - 2)\, 2x - ax\} = \vec{0}$

ie., $(a - 4)\, z^2\, j + (a - 4)\, x\, k = \vec{0}$

The above equation is identically satisfied when $a = 4$.

Now consider $\nabla\phi = (\vec{F})_{a = 4}$

ie.,
$$\frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = (4xy - z^3)\, i + 2x^2\, j - 3xz^2\, k$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = 4xy - z^3 \ \therefore\ \phi = \int (4xy - z^3)\, dx + f_1(y, z)$$

ie., $\phi = 2x^2 y - xz^3 + f_1(y, z)$ ...(1)

$$\frac{\partial\phi}{\partial y} = 2x^2 \ \therefore\ \phi = \int 2x^2\, dy + f_2(x, z)$$

ie., $\phi = 2x^2 y + f_2(x, z)$ ...(2)

$$\frac{\partial \phi}{\partial z} = -3xz^2 \quad \therefore \quad \phi = \int -3xz^2\, dz + f_3(x, y)$$

ie., $\phi = -xz^3 + f_3(x, y)$ ...(3)

Let us choose $f_1(y, z) = 0,\ f_2(x, z) = -xz^3,\ f_3(x, y) = 2x^2 y$ from (1), (2) & (3)

Thus $\phi = 2x^2 y - xz^3$

---

46. *Find constants a and b such that $\vec{F} = (axy + z^3)\, i + (3x^2 - z)\, j + (bxz^2 - y)\, k$ is irrotational. Also find a scalar function ϕ such that $\vec{F} = \nabla\phi$.*

>> We have to find a and b such that curl $\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (axy + z^3) & (3x^2 - z) & (bxz^2 - y) \end{vmatrix} = \vec{0}$$

ie., $i(-1+1) - j(bz^2 - 3z^2) + k(6x - ax) = \vec{0}$

ie., $-z^2(b-3)\, j + x(6-a)\, k = \vec{0}$

The above equation is identically satisfied when

$$b - 3 = 0 \text{ and } 6 - a = 0 \qquad \therefore \quad \mathbf{a = 6 \text{ and } b = 3}$$

Now consider $\nabla\phi = \vec{F}$ when $a = 6,\ b = 3$

ie., $\dfrac{\partial \phi}{\partial x} i + \dfrac{\partial \phi}{\partial y} j + \dfrac{\partial \phi}{\partial z} k = (6xy + z^3)\, i + (3x^2 - z)\, j + (3xz^2 - y)\, k$

$\Rightarrow$ $\dfrac{\partial \phi}{\partial x} = 6xy + z^3 \quad \therefore \quad \phi = \int (6xy + z^3)\, dx + f_1(y, z)$

ie., $\phi = 3x^2 y + xz^3 + f_1(y, z)$ ...(1)

$$\frac{\partial \phi}{\partial y} = (3x^2 - z) \quad \therefore \quad \phi = \int (3x^2 - z)\, dy + f_2(x, z)$$

ie., $\phi = 3x^2 y - yz + f_2(x, z)$ ...(2)

$$\frac{\partial \phi}{\partial z} = (3xz^2 - y) \quad \therefore \quad \phi = \int (3xz^2 - y)\, dz + f_3(x, y)$$

ie., $\phi = xz^3 - yz + f_3(x, y)$ $\quad\ldots(3)$

Let us choose $f_1(y, z) = -yz$, $f_2(x, z) = xz^3$, $f_3(x, y) = 3x^2y$ from (1), (2) & (3).

Thus the required $\phi = 3x^2y + xz^3 - yz$

---

47. *If* $\vec{F} = (x+y+az)\,i + (bx+2y-z)\,j + (x+cy+2z)\,k$ *find a, b, c such that curl* $F = \vec{0}$ *and then find* ϕ *such that* $\vec{F} = \nabla\phi$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x+y+az) & (bx+2y-z) & (x+cy+2z) \end{vmatrix} = \vec{0}$$

ie., $i(c+1) - j(1-a) + k(b-1) = \vec{0}$

$\Rightarrow$ $c+1 = 0,\ 1-a = 0,\ b-1 = 0$

$\therefore$ $a = 1,\ b = 1,\ c = -1$ are the required values.

Now consider $\nabla\phi = \vec{F}$ when $a = 1,\ b = 1,\ c = -1$.

ie., $\dfrac{\partial\phi}{\partial x}\,i + \dfrac{\partial\phi}{\partial y}\,j + \dfrac{\partial\phi}{\partial z}\,k$

$= (x+y+z)\,i + (x+2y-z)\,j + (x-y+2z)\,k$

$\Rightarrow$ $\dfrac{\partial\phi}{\partial x} = x+y+z \quad \therefore \quad \phi = \int (x+y+z)\,dx + f_1(y, z)$

ie., $\phi = \dfrac{x^2}{2} + xy + xz + f_1(y, z)$ $\quad\ldots(1)$

$\dfrac{\partial\phi}{\partial y} = x+2y-z \quad \therefore \quad \phi = \int (x+2y-z)\,dy + f_2(x, z)$

ie., $\phi = xy + y^2 - yz + f_2(x, z)$ $\quad\ldots(2)$

$\dfrac{\partial\phi}{\partial z} = x-y+2z \quad \therefore \quad \phi = \int (x-y+2z)\,dz + f_3(x, y)$

ie., $\phi = xz - yz + z^2 + f_3(x, y)$ $\quad\ldots(3)$

Let us choose $f_1(y, z) = y^2 - yz + z^2$,

$f_2(x, z) = \dfrac{x^2}{2} + xz + z^2$, $f_3(x, y) = \dfrac{x^2}{2} + xy + y^2$ from (1), (2) and (3)

Thus the required $\phi = \dfrac{x^2}{2} + xy + xz + y^2 - yz + z^2$

[illegible]

$$\vec{F} = 2xyz^2\, i + (x^2 z^2 + z\cos yz)\, j + (2x^2 yz + y\cos yz)\, k$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & (x^2 z^2 + z\cos yz) & (2x^2 yz + y\cos yz) \end{vmatrix}$$

$$i\,[\,2x^2 z + (-yz\sin yz + \cos yz) - 2x^2 z - (-yz\sin yz + \cos yz)\,]$$

$$-j\,(4xyz - 4xyz) + k\,(2xz^2 - 2xz^2) = \vec{0}$$

$\therefore$ **curl $\vec{F} = \vec{0} \Rightarrow \vec{F}$ is a potential field.**

Consider $\nabla\phi = \vec{F}$

ie., $$\frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = 2xyz^2\, i + (x^2 z^2 + z\cos yz)\, j + (2x^2 yz + y\cos yz)\, k$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = 2xyz^2 \qquad \therefore \quad \phi = \int 2xyz^2\, dx + f_1(y, z)$$

ie., $$\phi = x^2 y z^2 + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = x^2 z^2 + z\cos yz \qquad \phi = \int (x^2 z^2 + z\cos yz)\, dy + f_2(x, z)$$

ie., $$\phi = x^2 y z^2 + \sin yz + f_2(x, z) \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = 2x^2 yz + y\cos yz$$

$$\phi = \int (2x^2 yz + y\cos yz)\, dz + f_3(x, y)$$

ie., $$\phi = x^2 yz^2 + \sin yz + f_3(x, y) \qquad \ldots (3)$$

Let us choose $f_1(y, z) = \sin yz$, $f_2(x, z) = 0$, $f_3(x, y) = 0$ from (1), (2) and (3)

Thus the required $\phi = x^2 y z^2 + \sin yz$

49. *Show that* $\vec{F} = \dfrac{\vec{r}}{r^3}$ *is both solenoidal and irrotational*

>> We have $\vec{F} = r^{-3}\,\vec{r}$ and we need to show that div $\vec{F} = 0$, and curl $\vec{F} = \vec{0}$.

We have to proceed on the same lines as in Examples 33 and 34 to obtain the result. It is left as an *Exercise* to the reader to establish the results.

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50. *Find the values of* n *such that the vector* $r^n\,\vec{r}$ *is*
(a) solenoidal (b) irrotational (c) both solenoidal and irrotational

>> We need to first establish the result for div $(r^n\,\vec{r})$ and curl $(r^n\,\vec{r})$. These have been worked in Examples 33 and 34.

We have obtained

$$\nabla \cdot (r^n\,\vec{r}) = (n+3)\,r^n \qquad \ldots (1)$$

$$\nabla \times (r^n\,\vec{r}) = \vec{0} \qquad \ldots (2)$$

(a) $\nabla \cdot (r^n\,\vec{r}) = 0$ when $n = -3$ with reference to (1)

$\therefore$ $r^n\,\vec{r}$ **is solenoidal for** $n = -3$

(b) $\nabla \times (r^n\,\vec{r}) = \vec{0}$ with reference to (2)

$\therefore$ $r^n\,\vec{r}$ **is irrotational for all values of** n

(c) Combining these two cases we can easily conclude that,

$r^n\,\vec{r}$ **is both solenoidal and irrotational for** $n = -3$

4.16 Vector Identities

These are some properties relating to various meaningful combinations of *gradient, divergence, curl and laplacian*. These are established by taking a general scalar point function or a vector point function.

These have to be **remembered** for working certain types of problems.

V.I - 1 curl (grad ϕ) $= \vec{0}$, or $\nabla \times (\nabla\phi) = \vec{0}$

Proof : Let ϕ be a scalar point function of x, y, z. grad $\phi = \nabla\phi = \dfrac{\partial\phi}{\partial x}\,i + \dfrac{\partial\phi}{\partial y}\,j + \dfrac{\partial\phi}{\partial z}\,k$

$$\text{curl } (\text{grad } \phi) = \nabla \times (\nabla\phi) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}$$

ie, $$= \Sigma \left\{ \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right) \right\} i = \Sigma \left(\frac{\partial^2 \phi}{\partial y\, \partial z} - \frac{\partial^2 \phi}{\partial z\, \partial y} \right) i = \vec{0}$$

Thus **curl (grad ϕ) = 0**, for any scalar function ϕ

V.I - 2 div (curl $\vec{A}$) = 0 , or $\nabla \cdot (\nabla \times \vec{A}) = 0$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

Now **div (curl $\vec{A}$)** $= \nabla \cdot (\nabla \times \vec{A})$

$$= \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) = \Sigma \left(\frac{\partial^2 a_3}{\partial x\, \partial y} - \frac{\partial^2 a_2}{\partial x\, \partial z} \right)$$

On expanding we get,

$$\frac{\partial^2 a_3}{\partial x\, \partial y} - \frac{\partial^2 a_2}{\partial x\, \partial z} + \frac{\partial^2 a_1}{\partial y\, \partial z} - \frac{\partial^2 a_3}{\partial y\, \partial x} + \frac{\partial^2 a_2}{\partial z\, \partial x} - \frac{\partial^2 a_1}{\partial z\, \partial y} = 0$$

Thus **div (curl $\vec{A}$) = 0**, for any vector function $\vec{A}$

V.I - 3 curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$ or $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

Now curl (curl $\vec{A}$) = $\nabla \times (\nabla \times \vec{A})$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial a_3}{\partial y} - \dfrac{\partial a_2}{\partial z}\right) & \left(\dfrac{\partial a_1}{\partial z} - \dfrac{\partial a_3}{\partial x}\right) & \left(\dfrac{\partial a_2}{\partial x} - \dfrac{\partial a_1}{\partial y}\right) \end{vmatrix}$$

$$= \Sigma\, i \left\{ \frac{\partial}{\partial y}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) \right\}$$

$$= \Sigma\, i \left(\frac{\partial^2 a_2}{\partial y\, \partial x} + \frac{\partial^2 a_3}{\partial z\, \partial x}\right) - \Sigma\, i \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}\right), \text{ by rearranging.}$$

(In order to get ∇^2 in the second term and observing that we do not have the second order partial derivative w.r.t. x we must think of adding and subtracting the same)

Adding and subtracting $\Sigma\, i \dfrac{\partial^2 a_1}{\partial x^2}$ we get

$$\Sigma\, i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_2}{\partial y\, \partial x} + \frac{\partial^2 a_3}{\partial z\, \partial x}\right) - \Sigma\, i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}\right)$$

$$= \Sigma\, i \frac{\partial}{\partial x}\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) - \Sigma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) a_1\, i$$

$$= \Sigma \frac{\partial}{\partial x} (\text{div } \vec{A})\, i - \nabla^2\, \Sigma\, a_1\, i = \text{grad} (\text{div } \vec{A}) - \nabla^2 \vec{A}$$

Thus **curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$**

V.I - 4 div $(\phi \vec{A}) = \phi\,(\text{div } \vec{A}) + \text{grad } \phi \cdot \vec{A}$ or $\nabla \cdot (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + \nabla \phi \cdot \vec{A}$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z and ϕ be a scalar point function of x, y, z

$$\phi \vec{A} = \phi (a_1 i + a_2 j + a_3 k) = \Sigma\, (\phi a_1)\, i$$

Now div $(\phi \vec{A}) = \nabla \cdot (\phi \vec{A})$

$$\left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma (\phi a_1) i$$

$$= \Sigma \frac{\partial}{\partial x} (\phi a_1) = \Sigma \left(\phi \frac{\partial a_1}{\partial x} + \frac{\partial \phi}{\partial x} a_1 \right)$$

ie., $$\text{div } (\phi \vec{A}) = \phi \Sigma \frac{\partial a_1}{\partial x} + \Sigma \frac{\partial \phi}{\partial x} i \cdot \Sigma a_1 i$$

(**Note** *The second term which is of the form* $\Sigma A_1 B_1$ *is being written as* $\Sigma A_1 i \cdot \Sigma B_1 i$)

Thus $$\textbf{div } (\phi \vec{A}) = \phi (\textbf{div } \vec{A}) + \textbf{grad } \phi \cdot \vec{A}$$

VI - 5 curl $(\phi \vec{A}) = \phi$ (curl $\vec{A}$) + grad $\phi \times \vec{A}$ or

$\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$

Proof : Let ϕ and $\vec{A} = a_1 i + a_2 j + a_3 k$ be respectively scalar and vector point functions of x, y, z

$$\therefore \quad \phi \vec{A} = (\phi a_1) i + (\phi a_2) j + (\phi a_3) k$$

Now $$\text{curl } (\phi \vec{A}) = \nabla \times (\phi \vec{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix}$$

ie., $$= \Sigma i \left\{ \frac{\partial}{\partial y} (\phi a_3) - \frac{\partial}{\partial z} (\phi a_2) \right\}$$

$$= \Sigma i \left\{ \left(\phi \frac{\partial a_3}{\partial y} + \frac{\partial \phi}{\partial y} a_3 \right) - \left(\phi \frac{\partial a_2}{\partial z} + \frac{\partial \phi}{\partial z} a_2 \right) \right\}$$

$$= \phi \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) i + \Sigma \left(\frac{\partial \phi}{\partial y} a_3 - \frac{\partial \phi}{\partial z} a_2 \right) i$$

$$= \phi \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$$

Thus $\quad \text{curl}\,(\phi\,\vec{A}) = \phi\,(\text{curl}\,\vec{A}) + \nabla\phi \times \vec{A}$

Remark : *V.I - 4 and 5 when presented interms of ∇ are in a format similar to the product rule of differentiation.*

V.I - 6 $\text{div}\,(\vec{A} \times \vec{B}) = \vec{B} \cdot \text{curl}\,\vec{A} - \vec{A} \cdot \text{curl}\,\vec{B}$ or

$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

Proof Let $\vec{A} = a_1 i + a_2 j + a_3 k$ and $\vec{B} = b_1 i + b_2 j + b_3 k$, be two vector point functions of x, y, z

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \Sigma\, i\,(a_2 b_3 - a_3 b_2)$$

Now $\quad \text{div}\,(\vec{A} \times \vec{B}) = \nabla \cdot (\vec{A} \times \vec{B})$

$$= \left(\Sigma \frac{\partial}{\partial x} i\right) \cdot \Sigma\, i\,(a_2 b_3 - a_3 b_2)$$

$$= \Sigma \frac{\partial}{\partial x}(a_2 b_3 - a_3 b_2)$$

$$= \Sigma \left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x} \right)$$

On expanding we get

$$\left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x} \right) + \left(a_3 \frac{\partial b_1}{\partial y} + b_1 \frac{\partial a_3}{\partial y} - a_1 \frac{\partial b_3}{\partial y} - b_3 \frac{\partial a_1}{\partial y} \right)$$

$$+ \left(a_1 \frac{\partial b_2}{\partial z} + b_2 \frac{\partial a_1}{\partial z} - a_2 \frac{\partial b_1}{\partial z} - b_1 \frac{\partial a_2}{\partial z} \right)$$

(Note: *We have to focus our attention on the R.H.S. of the desired result and accordingly plan for rearranging the terms. Since we have dot products with $\vec{B}$ and $\vec{A}$, naturally we should have the first term lead by b_1 and second term lead by a_1 with the summation notation)*

i.e., $$= \Sigma\, b_1 \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) - \Sigma\, a_1 \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z} \right)$$

$$= (\Sigma\, b_1 i) \cdot \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) i - (\Sigma\, a_1 i) \cdot \Sigma \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z} \right) i$$

$$(\because \quad \Sigma A_1 B_1 = \Sigma A_1 i \cdot \Sigma B_1 i)$$

$$= (\Sigma b_1 i) \cdot \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} - (\Sigma a_1 i) \cdot \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Thus $\quad \mathbf{div}\ (\vec{A} \times \vec{B}) = \vec{B} \cdot \mathbf{curl}\ \vec{A} - \vec{A} \cdot \mathbf{curl}\ \vec{B}$

WORKED PROBLEMS

Note : *We work a few problems by using vector identities. Some problems can be done without using the vector identities but the same becomes very simple by the use of vector identities. In some of the problems we have to necessarily use the identities to arrive at the desired result*

51 *Show that the gradient of any scalar point function is irrotational and the curl of any vector point function is solenoidal*

>> Let $\phi(x, y, z)$ be the scalar point function and $\vec{A}(x, y, z)$ be the vector point function We have to show that grad ϕ is irrotational and curl $\vec{A}$ is solenoidal That is to prove that

$$\text{curl}\ (\text{grad}\ \phi) = 0 \text{ and } \text{div}\ (\text{curl}\ \vec{A}) = 0$$

We only need to establish the vector identities V I - 1 and V I - 2

52. *If $\phi(x, y, z)$ is a harmonic function, show that grad ϕ is both solenoidal and irrotational*

>> $\phi(x, y, z)$ is a harmonic function implies that it satisfies the Laplace's equation $\nabla^2 \phi = 0$.

$$\therefore \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

We have to show that div (grad ϕ) = 0 and curl (grad ϕ) = $\vec{0}$

$$\text{div}\ (\text{grad}\ \phi) = \nabla \cdot \nabla\phi = \left(\Sigma \frac{\partial}{\partial x} i\right) \cdot \left(\Sigma \frac{\partial \phi}{\partial x} i\right) = \Sigma \frac{\partial^2 \phi}{\partial x^2}$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

By using (1) we have $\text{div}\ (\text{grad}\ \phi) = 0$ and hence **grad ϕ is solenoidal.**

Also, establishing $\text{curl}\ (\text{grad}\ \phi) = \vec{0}$ is nothing but establishing V.I - 1

53. *Prove that if $\phi(x, y, z)$ is a scalar point function then $\phi\nabla\phi$ is irrotational*

>> We have to prove that $\text{curl}\ (\phi\nabla\phi) = \vec{0}$.

ie., to prove that $\nabla \times (\phi\nabla\phi) = \vec{0}$.

We have the vector identity (V.I - 5)

$$\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A} \qquad \textit{(can be assumed)}$$

Taking $\vec{A} = \nabla\phi$ we have

$$\nabla \times (\phi\nabla\phi) = \phi\nabla \times (\nabla\phi) + \nabla\phi \times \nabla\phi$$

$\nabla \times (\phi\nabla\phi) = \vec{0} + \vec{0} = \vec{0}$, because the first term is zero by the vector identity (V.I - 1) and the second term is zero since $\vec{V} \times \vec{V}$ is $\vec{0}$ for any vector $\vec{V}$

Thus $\nabla \times (\phi\nabla\phi) = \vec{0} \Rightarrow$ **$\phi\nabla\phi$ is irrotational.**

Aliter : *(Without using the vector identities)*

$$\nabla \times (\phi\nabla\phi) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\phi\dfrac{\partial\phi}{\partial x}\right) & \left(\phi\dfrac{\partial\phi}{\partial y}\right) & \left(\phi\dfrac{\partial\phi}{\partial z}\right) \end{vmatrix}$$

$$= \Sigma\, i\left\{\frac{\partial}{\partial y}\left(\phi\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\phi\frac{\partial\phi}{\partial y}\right)\right\}$$

$$= \Sigma\, i\left\{\left(\phi\frac{\partial^2\phi}{\partial y\,\partial z} + \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z} - \phi\frac{\partial^2\phi}{\partial z\,\partial y} - \frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial y}\right)\right\} = \vec{0}$$

$\phi\nabla\phi$ is irrotational.

54. *If u and v are scalar point functions prove that $\nabla\cdot(u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$. What happens when u and v are harmonic functions*

>> We have the vector identity (V.I - 4)

$$\nabla\cdot(\phi\vec{A}) = \phi(\nabla\cdot\vec{A}) + \nabla\phi\cdot\vec{A}$$

Taking $\phi = u$ and $\vec{A} = \nabla v$ we obtain

$$\nabla\cdot(u\nabla v) = u\{\nabla\cdot(\nabla v)\} + \nabla u\cdot\nabla v$$

ie., $$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u\cdot\nabla v \qquad \ldots(1)$$

Similarly we have,

$$\nabla \cdot (v \nabla u) = v \nabla^2 u + \nabla v \cdot \nabla u$$

$\therefore$ (1) − (2) will give us

$$\nabla \cdot (u \nabla v) - \nabla \cdot (v \nabla u) = u \nabla^2 v - v \nabla^2 u$$

Thus $\nabla \cdot (u \nabla v - v \nabla u) = u \nabla^2 v - v \nabla^2 u$

Further if u and v are harmonic functions then $\nabla^2 u = 0$, $\nabla^2 v = 0$

$\therefore$ (3) becomes $\nabla \cdot (u \nabla v - v \nabla u) = 0$

$\Rightarrow$ $u \nabla v - v \nabla u$ **is solenoidal.**

---

55. *If $\vec{F_1}$ and $\vec{F_2}$ are irrotational, prove that $\vec{F_1} \times \vec{F_2}$ is solenoidal*

>> $\vec{F_1}$ and $\vec{F_2}$ are irrotational by data.

$\Rightarrow$ curl $\vec{F_1} = \vec{0}$ and curl $\vec{F_2} = \vec{0}$ ... (1)

We have to prove that div $(\vec{F_1} \times \vec{F_2}) = 0$

We have the vector identity (V.I - 6),

div $\left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot$ curl $\vec{A} - \vec{A} \cdot$ curl $\vec{B}$ *(assumed)*

$\therefore$ div $(\vec{F_1} \times \vec{F_2}) = \vec{F_2} \cdot$ curl $\vec{F_1} - \vec{F_1} \cdot$ curl $\vec{F_2}$

ie., div $(\vec{F_1} \times \vec{F_2}) = \vec{F_2} \cdot \vec{0} - \vec{F_1} \cdot \vec{0} = 0$, by using (1)

$\therefore$ div $(\vec{F_1} \times \vec{F_2}) = 0 \Rightarrow \vec{F_1} \times \vec{F_2}$ **is solenoidal.**

---

56. *If $u \vec{F} = \nabla v$, prove that $\vec{F}$ and curl $\vec{F}$ are at right angles*

>> $\vec{F} = \dfrac{1}{u} \nabla v$ and we have to prove that $\vec{F} \cdot$ curl $\vec{F} = 0$

We shall first find curl $\vec{F}$ where $\vec{F}$ is of the form $\phi \vec{A}$
Let us consider the vector identity (V.I - 5)

$$\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$$

$$\therefore \quad \nabla \times \left(\frac{1}{u} \nabla v\right) = \frac{1}{u} \{\nabla \times (\nabla v)\} + \nabla \left(\frac{1}{u}\right) \times \nabla v$$

The first term in the RHS of this equation is zero by a vector identity curl (grad ϕ) = 0 (V.I - 1)

$\therefore \qquad \nabla \times \left(\frac{1}{u} \nabla v \right) = \nabla \left(\frac{1}{u} \right) \times \nabla v \quad$ ie., curl $\vec{F} = \nabla \left(\frac{1}{u} \right) \times \nabla v$

Now $\quad \vec{F} \cdot \text{curl } \vec{F} = \left(\frac{1}{u} \nabla v \right) \cdot \left\{ \nabla \left(\frac{1}{u} \right) \times \nabla v \right\}$

R.H.S of this equation is a scalar triple product or the box product of three vectors

ie., $\qquad \vec{F} \cdot \text{curl } \vec{F} = \left[\frac{1}{u} \nabla v, \nabla \left(\frac{1}{u} \right), \nabla v \right]$

ie., $\qquad \vec{F} \cdot \text{curl } \vec{F} = \frac{1}{u} \left[\nabla v, \nabla \left(\frac{1}{u} \right), \nabla v \right] = 0$

(**Note** : *Since the box product of three vectors is equal to coefficient determinant, we have removed* $\frac{1}{u}$ *as a commmon factor from the first row*)

Hence $\vec{F} \cdot \text{curl } \vec{F} = 0$, since two vectors are identical in the box product

Thus $\vec{F}$ **is perpendicular to curl** $\vec{F}$

---

57. *By using vector identities prove that*

(*a*) $\quad \nabla \cdot (r^n \vec{r}) = (n+3)\, r^n \qquad$ (*b*) $\quad \nabla \times (r^n \vec{r}) = \vec{0}$

>> (a) We have $\nabla \cdot (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + \nabla \phi \cdot \vec{A}$

$$\therefore \qquad \nabla \cdot (r^n \vec{r}) = r^n (\nabla \cdot \vec{r}) + \nabla r^n \cdot \vec{r} \qquad \ldots (1)$$

Now $\quad \nabla \cdot \vec{r} = \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot (\Sigma\, x\, i) = 1+1+1 = 3$

Also $\quad \nabla r^n = n r^{n-2} \vec{r} \qquad$ *(Refer Example - 32)*

$$\nabla r^n \cdot \vec{r} = n r^{n-2} (\vec{r} \cdot \vec{r}) = n r^{n-2} \,|\vec{r}|^2 = n r^{n-2} r^2 = n r^n$$

Using these in the R.H.S of (1) we get,

$$\nabla \cdot (r^n \vec{r}) = 3 r^n + n r^n$$

Thus $\quad \boldsymbol{\nabla \cdot (r^n \vec{r}) = (n+3)\, r^n}$

(b) We have $\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$

$$\therefore \qquad \nabla \times (r^n \vec{r}) = r^n (\nabla \times \vec{r}) + \nabla r^n \times \vec{r} \qquad \ldots (2)$$

60. *Prove that* $\nabla^2 (r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$

>> $\nabla^2 (r^n \vec{r})$ involves Laplacian of a vector point function and hence we recall the identity

$$\text{curl}\,(\text{curl}\,\vec{A}) = \text{grad}\,(\text{div}\,\vec{A}) - \nabla^2 \vec{A} \qquad \ldots (\text{V.I - 3})$$

$$\therefore \quad \nabla^2 \vec{A} = \text{grad}\,(\text{div}\,\vec{A}) - \text{curl}\,(\text{curl}\,\vec{A})$$

Now $\nabla^2 (r^n \vec{r}) = \text{grad}\,\{\text{div}\,(r^n \vec{r})\} - \text{curl}\,\{\text{curl}\,(r^n \vec{r})\}$

But $\text{div}\,(r^n \vec{r}) = (n+3)\, r^n$ ... *Refer Example - 33*

and $\text{curl}\,(r^n \vec{r}) = \vec{0}$ ... *Refer Example - 34*

$\therefore \quad \nabla^2 (r^n \vec{r}) = \text{grad}\,\{(n+3)\, r^n\} = (n+3)\,\text{grad}\,(r^n)$

But $\text{grad}\,(r^n) = n\, r^{n-2}\, \vec{r}$ ... *Refer Example - 32*

Thus $\nabla^2 (r^n \vec{r}) = (n+3)\, n\, r^{n-2} \vec{r} = n(n+3)\, r^{n-2} \vec{r}$

EXERCISES

Find the unit vector normal to the following surfaces at the indicated points (1 to 3)

1. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$.
2. $x^2 y + y^2 z + z^2 x = 5$ at $(1, -1, 2)$.
3. $xy + yz + zx = 1$ at $(-1, 2, 3)$

Find the directional derivatives of the following : (4 to 9)

4. $\phi = xy + yz + zx$ at $(1, 2, 3)$ along $3i + 4j + 5k$.
5. $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ along $i + 2j + 2k$.
6. $\phi = e^{2x - y + z}$ at $(1, 1, -1)$ in the direction towards the point $(3, 5, -2)$
7. $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ along the (a) x-axis (b) direction which makes equal angles with the coordinate axes (c) normal to the surface $xy + yz + zx = 3$ at the point $(1, 1, 1)$.
8. $\phi = x^2 y + y^2 z + z^2 x$ at $(1, 1, 1)$ along the tangent to the curve $\vec{r} = t\,i + t^2 j + t^3 k$.
9. $\phi = e^x \cos yz$ at the origin in the direction of the tangent to the curve $x = a \sin t$, $y = a \cos t$, $z = at$ at $t = \pi/4$

10. In which direction the directional derivative of the function $x^2 y^2 z^3$ is maximum at the point $(1, -1, 2)$? Find the magnitude of this maximum.

11. If the directional derivative of the function $\phi = axy^2 + byz + cz^2 x^3$ at $(1, 2, -1)$ has a maximum magnitude of 64 units in the direction parallel to the z-axis show that the values of a, b, c satisfy the equation $a + b + c = 22$.

12. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y + z = 2$ at the point $(1, 1, 1)$ and also find the angle between the normals to the surface $x \log z = y^2 - 1$ at the points $(1, 1, 1)$ and $(2, 1, 1)$.

13. Show that the surfaces $4x^2 + z^3 = 4$ and $5x^2 - 2yz - 9x = 0$ intersect each other orthogonally at the point $(1, -1, 2)$.

14. Find the value of the constants a and b such that the surface $x^2 + ayz = 3x$ and $bx^2 y + z^3 = (b - 8)\, y$ intersect each other at right angles at the point $(1, 1, -2)$.

15. Find grad (div $\vec{A}$) and div (curl $\vec{A}$) for the vector $\vec{A} = x^2 i + 3yj + x^3 k$

16. If $\vec{A} = x y^2 i + 2x^2 yz\, j - 3y^2 z\, k$ find div $\vec{A}$, curl $\vec{A}$ and div (curl $\vec{A}$) at the point $(2, 1, 1)$.

17. If $\vec{A} = \Sigma\, (y^2 + z^2 - x^2)\, i$ prove that div $\vec{A} = -2\, \Sigma\, x$, curl $\vec{A} = 2\, \Sigma\, (y - z)\, i$.

18. If $\phi_1 = x + y + z$, $\phi_2 = x^2 + y^2 + z^2$, $\phi_3 = xy + yz + zx$ prove that the scalar triple product of the vectors $\nabla\phi_1$, $\nabla\phi_2$, $\nabla\phi_3$ is zero.

19. If $\vec{A} = x^2 y\, i - 2xz\, j + 2yz\, k$, find curl (curl $\vec{A}$) and verify that curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$.

20. If $\phi = xyz$ and $\vec{A} = x^2 yz\, i + y^2 zx\, j + z^2 xy\, k$ verify that

$$\nabla \cdot (\phi \vec{A}) = \phi\, (\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$$

21. If $\vec{A} = \Sigma\, x^2 i$, $\vec{B} = \Sigma\, yz\, i$ verify that

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) = 0.$$

Prove the following

22. $\nabla \cdot \left\{ \vec{r}\, \nabla \left(\dfrac{1}{r^3} \right) \right\} = \dfrac{3}{r^4}$

23. $\nabla \left\{ \nabla \cdot \left(\dfrac{\vec{r}}{r} \right) \right\} = -\dfrac{2\vec{r}}{r^3}$

24. $\nabla^2 (\log r) = \frac{1}{r^2}$

25. $\nabla^2 \left(\frac{x}{r^3} \right) = 0$

26. $\nabla^2 (fg) = f\, \nabla^2 g + g\, \nabla^2 f + 2\, \nabla f \cdot \nabla g$

27. Find the value of the constant 'a' such that the vector function $\overrightarrow{A} = y(ax^2 + z)\, i + x(y^2 - z^2)\, j + 2xy(z - xy)\, k$ is solenoidal. For this value of 'a' show that curl $\overrightarrow{A}$ is also solenoidal.

28. Show that the following vector field is irrotational.
$\overrightarrow{F} = (\sin y + z)\, i + (x \cos y - z)\, j + (x - y)\, k$
Also find the scalar function ϕ such that $\overrightarrow{F} = \nabla\phi$

29. Find the values of the constants a, b, c such that
$\overrightarrow{F} = (x + 2y + az)\, i + (bx - 3y - z)\, j + (4x + cy + 2z)\, k$ is conservative.
Also find its scalar potential.

30. Prove that $\nabla\phi_1 \times \nabla\phi_2$ is irrotational.

ANSWERS

1. $\frac{-i + 3j + 2k}{\sqrt{14}}$

2. $\frac{2i - 3j + 5k}{\sqrt{38}}$

3. $\frac{5i + 2j + k}{\sqrt{30}}$

$\frac{46}{5\sqrt{2}}$

5. $-\frac{11}{3}$

6. $\frac{-e^2}{\sqrt{21}}$

7. (a) 1 (b) and (c) : $\frac{-5}{\sqrt{3}}$

8. $\frac{18}{\sqrt{14}}$

9. 1

10. $4(4i - 4j + 3j)$, $4\sqrt{41}$

11. $a = 6,\ b = 24,\ c = -8$

12. $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$, $\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

14. $a = -1,\ b = 2$

15. $2i$, 0

16. 6, $-11i + 4j$, 0

19. $2(x + 1)\, j$

27. $a = -2$

28. $\phi = x \sin y + xz - yz$

29. $a = 4,\ b = 2,\ c = -1$; $\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2$

4.2 Orthogonal Curvilinear Coordinates (O.C.C.)

4.21 Definitions

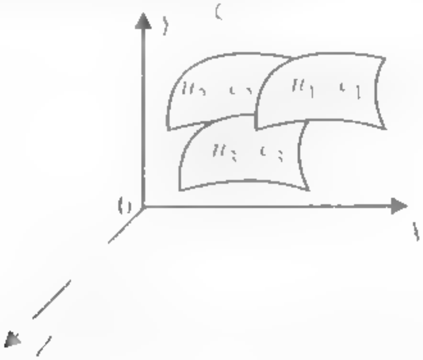
Curvilinear coordinates, curvilinear coordinate surfaces and curvilinear coordinate curves

Let the coordinates of any point P in space be (x, y, z) in the cartesian system. Suppose x, y, z are expressible interms of new coordinates (u_1, u_2, u_3), we can say that x, y, z are functions of u_1, u_2, u_3. Let us suppose that we are also in a position to express u_1, u_2, u_3 in terms of x, y, z by solving /eliminating. Then the coordinates (u_1, u_2, u_3) are known as ***curvilinear coordinates*** of the point P, where it is assumed that the correspondence between (x, y, z) and (u_1, u_2, u_3) is unique.

The surfaces $u_1 = c_1$ and $u_2 = c_2$, $u_3 = c_3$, c_1, c_2, c_3 being constants, are called ***coordinate surfaces*** and the intersection of each pair of these surfaces give rise to curves called ***coordinate curves.***

Orthogonal curvilinear coordinates

A system of curvilinear coordinates is said to be ***orthogonal*** if at each point the tangents to the coordinate curves are mutually perpendicular.



Unit Vectors, scale factors and orthogonality condition

Suppose $\vec{r} = x\,i + y\,j + z\,k$ be the position vector of a point in space, we have $\vec{r} = \vec{r}(u_1, u_2, u_3)$.

$\dfrac{\partial \vec{r}}{\partial u_1}, \dfrac{\partial \vec{r}}{\partial u_2}, \dfrac{\partial \vec{r}}{\partial u_3}$ are called the ***tangent vectors*** to the coordinate curves and the ***unit tangent vectors*** in the same direction are respectively

$$\hat{e}_1 = \frac{\partial \vec{r}}{\partial u_1} \Big/ \left| \frac{\partial \vec{r}}{\partial u_1} \right|, \quad \hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2} \Big/ \left| \frac{\partial \vec{r}}{\partial u_2} \right|, \quad \hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3} \Big/ \left| \frac{\partial \vec{r}}{\partial u_3} \right|$$

The quantities $h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right|$, $h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right|$, $h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right|$ are called *scale factors.*

For the ***orthogonality of the curvilinear coordinate system we must have***

$$\hat{e}_1 \cdot \hat{e}_2 = 0, \quad \hat{e}_2 \cdot \hat{e}_3 = 0, \quad \hat{e}_3 \cdot \hat{e}_1 = 0.$$

These are analogous to the property of basic unit vectors in the cartesian system

$$i \cdot j = 0, \quad j \cdot k = 0, \quad k \cdot i = 0$$

We have $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$, $\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$.

Thus $\hat{e}_1, \hat{e}_2, \hat{e}_3$ form a right handed system of vectors. If $\vec{A}$ is any vector in the orthogonal curvilinear coordinate system then

$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ where A_1, A_2, A_3 are scalar functions of u_1, u_2, u_3

In addition to the well acquainted rectangular cartesian coordinates (x, y, z) we introduce two new set of coordinates.

(i) ***Cylindrical polar coordinates*** (ρ, ϕ, z) given by the transformation

$$x = \rho \cos \phi, \ y = \rho \sin \phi, \ z = z$$

(ii) ***Spherical polar coordinates*** (r, θ, ϕ) given by the transformation

$$x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta.$$

We are familiar with the vector differential operator $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

and the Laplacian operator $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ operated on scalar and vector point functions.

If ψ is a scalar function and $\vec{A}$ is a vector function of x, y, z we know that

$$\nabla \psi = \text{grad}\, \psi, \ \nabla \cdot \vec{A} = \text{div}\, \vec{A}, \nabla \times \vec{A} = \text{curl}\, \vec{A}, \ \nabla^2 \psi = \text{Laplacian of } \psi$$

In this topic we obtain expressions for these in a general curvilinear coordinate system with special reference to the cylindrical system and spherical system as particular cases.

4.22 Scale factors of the cylindrical and spherical systems

Cylindrical system

The cylindrical polar coordinates (ρ, ϕ, z) is regarded as a particular case of the general orthogonal curvilinear coordinates (u_1, u_2, u_3) by setting $(u_1, u_2, u_3) = (\rho, \phi, z)$ and are related to the cartesian coordinates (x, y, z) by the transformation :

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z.$$

Thus $\vec{r} = x\,i + y\,j + z\,k$ becomes

$$\vec{r} = \rho\cos\phi\, i + \rho\sin\phi\, j + zk$$

We have by the definition of scale factors,

$$h_1 = \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \left|\cos\phi\, i + \sin\phi\, j + 0k\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left|-\rho\sin\phi\, i + \rho\cos\phi\, j + 0k\right| = \sqrt{\rho^2(\sin^2\phi + \cos^2\phi)} = \rho$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial z}\right| = \left|0i + 0j + 1k\right| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

Thus $h_1 = 1,\ h_2 = \rho,\ h_3 = 1$ for the cylindrical system.

Spherical system

We have $(u_1, u_2, u_3) = (r, \theta, \phi)$ and by the transformation $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ we have

$$\vec{r} = r\sin\theta\cos\phi\, i + r\sin\theta\sin\phi\, j + r\cos\theta\, k$$

$$h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = \left|\sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k\right|$$

i.e., $$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \left|r\cos\theta\cos\phi\, i + r\cos\theta\sin\phi\, j - r\sin\theta\, k\right|$$

i.e., $$= \sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left|-r\sin\theta\sin\phi\, i + r\sin\theta\cos\phi\, j + 0k\right|$$

ie., $= \sqrt{r^2 \sin^2\theta\,(\sin^2\phi + \cos^2\phi)} = r\sin\theta$

Thus $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ for the spherical system

Note : *$\vec{r} = xi + yj + zk$ will give us $h_1 = 1 = h_2 = h_3$ for the cartesian system*

4.23 Orthogonality of the cylindrical system

We have for the cylindrical system $\vec{r} = \rho\cos\phi\, i + \rho\sin\phi\, j + z\, k$

Let $\hat{e}_\rho,\ \hat{e}_\phi,\ \hat{e}_z$ be the basic unit vectors of this system

They are given by

$$\hat{e}_\rho = \frac{\partial \vec{r}}{\partial \rho} \Big/ \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \rho} = (\cos\phi\, i + \sin\phi\, j + 0k), \quad \text{since } h_1 = 1$$

$$\hat{e}_\phi = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho}(-\rho\sin\phi\, i + \rho\cos\phi\, j + 0k), \quad \text{since } h_2 = 1.$$

ie., $\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j + 0k$

$$\hat{e}_z = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \frac{1}{1}(0i + 0j + 1k) = 0i + 0j + 1k, \quad \text{since } h_3 = 1.$$

Now $\hat{e}_\rho \cdot \hat{e}_\phi = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0$, $\hat{e}_\phi \cdot \hat{e}_z = 0$, $\hat{e}_z \cdot \hat{e}_\rho = 0$

Thus the cylindrical system is orthogonal.

4.24 Orthogonality of the spherical system

We have for the spherical system

$$\vec{r} = r\sin\theta\cos\phi\, i + r\sin\theta\sin\phi\, j + r\cos\theta\, k$$

and let $\hat{e}_r,\ \hat{e}_\theta,\ \hat{e}_\phi$ be the basic unit vectors of this system

Further we have $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

Now, $\hat{e}_r = \dfrac{1}{h_1}\dfrac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r}(r\cos\theta\cos\phi\, i + r\cos\theta\sin\phi\, j - r\sin\theta\, k)$$

ie., $\hat{e}_\theta = \cos\theta\cos\phi\, i + \cos\theta\sin\phi\, j - \sin\theta\, k$

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r\sin\theta}(-r\sin\theta\sin\phi\, i + r\sin\theta\cos\phi\, j + 0k)$$

ie., $\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j + 0k$

Now $\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta = 0$

$$\hat{e}_\theta \cdot \hat{e}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$

$$\hat{e}_\phi \cdot \hat{e}_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\cos\phi\sin\phi = 0$$

Thus the spherical system is orthogonal.

4.25 | Arc length and volume element in the orthogonal curvilinear coordinate system

We have $\vec{r} = \vec{r}(u_1, u_2, u_3)$

$\therefore$ $$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1}du_1 + \frac{\partial \vec{r}}{\partial u_2}du_2 + \frac{\partial \vec{r}}{\partial u_3}du_3 \quad \textit{(total derivative)}$$

ie., $d\vec{r} = h_1\, du_1\, \hat{e}_1 + h_2\, du_2\, \hat{e}_2 + h_3\, du_3\, \hat{e}_3$

For a curve in space through the point P the *arc length* ds is given by the relation $ds = |\vec{dr}|$

ie., $ds = \sqrt{h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2}$

or $ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2$

The vector $\vec{dr}$ along the u_1 curve for which u_2 and u_3 are constants is given by $\vec{dr} = h_1\, du_1\, \hat{e}_1$ since $du_2 = 0 = du_3$.

Similarly along the u_2 curve and u_3 curve we have $h_2\, du_2\, \hat{e}_2$ and $h_3\, du_3\, \hat{e}_3$ respectively.

The volume of the rectangular parallelopiped formed by these is called the volume element dV at P in the orthogonal curvilinear coordinate system. Using the geometrical meaning of the scalar triple product of vectors we have,

$$dV = h_1\, du_1\, \hat{e}_1 \cdot (h_2\, du_2\, \hat{e}_2 \times h_3\, du_3\, \hat{e}_3) = h_1 h_2 h_3\, du_1\, du_2\, du_3\; \hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3)$$

But $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$ and $\hat{e}_1 \cdot \hat{e}_1 = 1$

$\therefore$ $dV = h_1 h_2 h_3\, du_1\, du_2\, du_3$

Thus $ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2$ and $dV = h_1 h_2 h_3\, du_1\, du_2\, du_3$

rollary

In the cylindrical system :

$$(u_1, u_2, u_3) = (\rho, \phi, z) \text{ and } h_1 = 1, h_2 = \rho, h_3 = 1$$

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \; ; \; dV = \rho \, d\rho \, d\phi \, dz$$

In the spherical system . $(u_1, u_2, u_3) = (r, \theta, \phi)$ and

$$h_1 = 1, h_2 = r, h_3 = r \sin\theta.$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \; dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

26 | Expression for $\nabla \psi$ in orthogonal curvilinear coordinates

nsider a scalar point function $\psi(u_1, u_2, u_3)$

$$\nabla \psi = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \qquad \ldots (1)$$

ere a_1, a_2, a_3 are to be determined.

also have $\vec{r} = \vec{r}(u_1, u_2, u_3)$ and as a total derivative,

$$\vec{dr} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$\vec{dr} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \qquad \ldots (2)$$

have the fact that x, y, z are functions of u_1, u_2, u_3 and vice-versa.

are also familiar with the result , $d\psi = \vec{dr} \cdot \nabla \psi$

ce $d\psi = a_1 h_1 du_1 + a_2 h_2 du_2 + a_3 h_3 du_3 \qquad \ldots (3)$

as a total derivative we also have from $\psi = \psi(u_1, u_2, u_3)$

$$d\psi = \frac{\partial \psi}{\partial u_1} du_1 + \frac{\partial \psi}{\partial u_2} du_2 + \frac{\partial \psi}{\partial u_3} du_3 \qquad \ldots (4)$$

ating the R.H.S of (3) and (4) we have,

$$a_1 h_1 = \frac{\partial \psi}{\partial u_1}, \; a_2 h_2 = \frac{\partial \psi}{\partial u_2}, \; a_3 h_3 = \frac{\partial \psi}{\partial u_3}$$

$$a_1 = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1}, \; a_2 = \frac{1}{h_2} \frac{\partial \psi}{\partial u_2}, \; a_3 = \frac{1}{h_3} \frac{\partial \psi}{\partial u_3}$$

Substituting these values in (1) we obtain,

$$\nabla \psi = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\hat{e}_3 = \Sigma \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1$$

Corollary

(i) In the cylindrical system :

$$(u_1, u_2, u_3) = (\rho, \phi, z);\ h_1 = 1\ h_2 = \rho,\ h_3 = 1$$

$$\nabla \psi = \frac{\partial \psi}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial \psi}{\partial \phi}\hat{e}_\phi + \frac{\partial \psi}{\partial z}\hat{e}_z$$

(ii) In the spherical system :

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{e}_\phi$$

4.27 Expression for div $\vec{A}$ in orthogonal curvilinear coordinates

Note : *The following vector identities will be useful.*

V.I - 1. $\nabla \times (\nabla \phi) = \vec{0}$

V.I - 2. $\nabla \cdot (\nabla \times \vec{A}) = 0$

V.I - 3. $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + (\nabla \phi \cdot \vec{A})$

V.I - 4. $\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + (\nabla \phi \times \vec{A})$

V.I - 5. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

Let $\quad \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$

$$\text{div}\,\vec{A} = \nabla \cdot \vec{A} = \nabla \cdot (A_1 \hat{e}_1) + \nabla \cdot (A_2 \hat{e}_2) + \nabla \cdot (A_3 \hat{e}_3) \quad \ldots (1)$$

We have $\nabla \psi = \Sigma \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 \quad \ldots (2)$

$$\therefore \quad \nabla u_1 = \frac{1}{h_1}\hat{e}_1 + 0 + 0$$

That is $\nabla u_1 = \frac{\hat{e}_1}{h_1},\ \nabla u_2 = \frac{\hat{e}_2}{h_2},\ \nabla u_3 = \frac{\hat{e}_3}{h_3}$

Also $\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = (h_2 \nabla u_2) \times (h_3 \nabla u_3)$

or $\quad \hat{e}_1 = h_2 h_3 (\nabla u_2 \times \nabla u_3) \quad \ldots (3)$

Let us consider only the first term in the R.H.S of (1) and proceed as follows

$$\nabla \cdot (A_1 \hat{e}_1) = \nabla \cdot \left\{ A_1 h_2 h_3 (\nabla u_2 \times \nabla u_3) \right\}, \text{ by using (3).}$$

ie., $= \nabla \cdot (\phi \vec{a})$ *where* $\phi = A_1 h_2 h_3, \ \vec{a} = (\nabla u_2 \times \nabla u_3)$

$$= \phi (\nabla \cdot \vec{a}) + \vec{a} \cdot \nabla \phi \qquad \text{[By using V.I - 3]}$$

$$= A_1 h_2 h_3 \left\{ \nabla \cdot (\nabla u_2 \times \nabla u_3) \right\} + (\nabla u_2 \times \nabla u_3) \cdot \nabla (A_1 h_2 h_3)$$

$$= A_1 h_2 h_3 \left\{ \nabla u_3 \cdot \nabla \times (\nabla u_2) - \nabla u_2 \cdot \nabla \times (\nabla u_3) \right\} + \frac{\hat{e}_1}{h_2 h_3} \cdot \nabla (A_1 h_2 h_3)$$

[*We have used V.I - 5 for the first term and expression* (3) *for the second term*]

$$= 0 + \frac{\hat{e}_1}{h_2 h_3} \cdot \Sigma \frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_2 h_3) \hat{e}_1$$

[*We have used V.I - 1 for the first term and the expression format of* $\nabla \psi$ *for the second term*]

By expanding R.H.S and taking the dot product we get,

$$\nabla \cdot (A_1 \hat{e}_1) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

$$\because \quad \hat{e}_1 \cdot \hat{e}_1 = 1, \ \hat{e}_1 \cdot \hat{e}_2 = 0, \ \hat{e}_1 \cdot \hat{e}_3 = 0$$

Similarly $\quad \nabla \cdot (A_2 \hat{e}_2) = \dfrac{1}{h_2 h_3 h_1} \dfrac{\partial}{\partial u_2} (A_2 h_3 h_1)$

$$\nabla \cdot (A_3 \hat{e}_3) = \frac{1}{h_3 h_1 h_2} \frac{\partial}{\partial u_3} (A_3 h_1 h_2)$$

Adding these results we have,

$$\nabla \cdot (A_1 \hat{e}_1) + \nabla \cdot (A_2 \hat{e}_2) + \nabla \cdot (A_3 \hat{e}_3) = \frac{1}{h_1 h_2 h_3} \cdot \Sigma \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

Using (1) for the L.H.S. we have,

$$\nabla \cdot \vec{A} = \text{div}\, \vec{A} = \frac{1}{h_1 h_2 h_3} \Sigma \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

Corollary

We deduce expressions for $\nabla \cdot \vec{A}$ in the cylindrical and spherical systems by using the expression for the same in the expanded form :

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right\}$$

We have $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$, for the cylindrical system. Hence we obtain,

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}(\rho A_1) + \frac{\partial}{\partial \phi}(A_2) + \frac{\partial}{\partial z}(\rho A_3)\right\} \qquad \textit{(Cylindrical system)}$$

Next, $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$ for the spherical system. Hence we obtain,

$$\nabla \cdot \vec{A} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, A_1) + \frac{\partial}{\partial \theta}(r\sin\theta\, A_2) + \frac{\partial}{\partial \phi}(r A_3)\right\} \qquad \textit{(Spherical system)}$$

1.78 Expression for curl $\vec{A}$ in orthogonal curvilinear coordinates

Let $\quad \vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$

$$\text{curl}\,\vec{A} = \nabla\times\vec{A} = \nabla\times(A_1\hat{e}_1) + \nabla\times(A_2\hat{e}_2) + \nabla\times(A_3\hat{e}_3) \qquad \ldots (1)$$

We have, $\nabla\psi = \Sigma\frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1 \quad \therefore \quad \nabla u_1 = \frac{1}{h_1}\hat{e}_1$

ie., $\quad \hat{e}_1 = h_1\nabla u_1 \qquad \ldots (2)$

We shall consider only the first term in the R.H.S. of (1) and proceed as follows

$\nabla\times(A_1\hat{e}_1) = \nabla\times(A_1 h_1\nabla u_1)$, by using (2).

ie., $\quad = \nabla\times(\phi\vec{a})$ *where* $\phi = A_1 h_1$, $\vec{a} = \nabla u_1$

$= \phi(\nabla\times\vec{a}) + \nabla\phi\times\vec{a} \qquad$ *[By using V.I - 4]*

$= A_1 h_1\{\nabla\times(\nabla u_1)\} + \nabla(A_1 h_1)\times\nabla u_1$

$= 0 + \nabla(A_1 h_1)\times\nabla u_1 \qquad \because \nabla\times\nabla\phi = 0$

$$= \left\{\frac{1}{h_1}\frac{\partial}{\partial u_1}(A_1 h_1)\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}(A_1 h_1)\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}(A_1 h_1)\hat{e}_3\right\}\times\left(\frac{\hat{e}_1}{h_1}\right)$$

where we have used the expression format of $\nabla \psi$ in the expanded form and (2)
Also, using the fact that

$\hat{e}_1 \times \hat{e}_1 = 0,\ \hat{e}_2 \times \hat{e}_1 = -\hat{e}_3\ ,\ \hat{e}_3 \times \hat{e}_1 = \hat{e}_2$ we have

$$\nabla \times (A_1 \hat{e}_1) = \frac{-\hat{e}_3}{h_1 h_2} \frac{\partial}{\partial u_2}(A_1 h_1) + \frac{\hat{e}_2}{h_1 h_3} \frac{\partial}{\partial u_3}(A_1 h_1)$$

Similarly by symmetry,

$$\nabla \times (A_2 \hat{e}_2) = \frac{-\hat{e}_1}{h_2 h_3} \frac{\partial}{\partial u_3}(A_2 h_2) + \frac{\hat{e}_3}{h_2 h_1} \frac{\partial}{\partial u_1}(A_2 h_2)$$

$$\nabla \times (A_3 \hat{e}_3) = \frac{-\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u_1}(A_3 h_3) + \frac{\hat{e}_1}{h_3 h_2} \frac{\partial}{\partial u_2}(A_3 h_3)$$

Adding these results, L.H.S becomes $\nabla \times \vec{A}$ according to (1) and R.H.S can be put in the determinant form as follows.

Thus $$\nabla \times \vec{A} = \frac{1}{h_1\ h_2\ h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

Corollary

(1) In the cylindrical system : $(u_1, u_2, u_3) = (\rho, \phi, z)$;
$h_1 = 1,\ h_2 = \rho,\ h_3 = 1$ and the basic unit vectors are denoted by $\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z$

$$\therefore \qquad \nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \rho \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_1 & \rho A_2 & A_3 \end{vmatrix}$$

(ii) In the spherical system : $(u_1, u_2, u_3) = (r, \theta, \phi)$;
$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ and the basic unit vectors are denoted by $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$

$$\therefore \qquad \nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\,\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_1 & r\,A_2 & r\sin\theta\,A_3 \end{vmatrix}$$

4.29 Expression for $\nabla^2 \psi$ (Laplacian of ψ) in orthogonal curvilinear coordinates

We know that $\nabla^2 \psi = \nabla \cdot \nabla \psi$ and we have

$$\nabla \psi = \Sigma \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 \qquad \ldots (1)$$

Also if $\vec{A} = A_1\,\hat{e}_1 + A_2\,\hat{e}_2 + A_3\,\hat{e}_3$ we have,

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \Sigma \frac{\partial}{\partial u_1} (A_1 h_2 h_3) \qquad \ldots (2)$$

We need to substitute (1) in (2). That is by taking $\vec{A} = \nabla \psi$ which is equivalent to taking $A_1 = \dfrac{1}{h_1} \dfrac{\partial \psi}{\partial u_1}$ since $\vec{A} = \Sigma A_1 \hat{e}_1$

$$\therefore \qquad \nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \Sigma \frac{\partial}{\partial u_1} \left(\frac{1}{h_1} \frac{\partial \psi}{\partial u_1} h_2 h_3 \right)$$

ie., $$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \Sigma \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) \qquad (\ )$$

Thus $$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right]$$

Corollary

$\nabla^2 \psi$ in the cylindrical system

In the cylindrical system we have $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1, h_2 = \rho, h_3 = 1$.

$$\therefore \quad \nabla^2 \psi = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\left[\rho \frac{\partial^2 \psi}{\partial \rho^2} + \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi^2} + \rho \frac{\partial^2 \psi}{\partial z^2}\right]$$

Thus $$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$\nabla^2 \psi$ in the spherical system

In the spherical system we have $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1, h_2 = r, h_3 = r \sin\theta$. Substituting in the general expression for $\nabla^2 \psi$ we get,

$$\nabla^2 \psi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[\sin\theta\left(r^2 \frac{\partial^2 \psi}{\partial r^2} + 2r\frac{\partial \psi}{\partial r}\right) + \sin\theta \frac{\partial^2 \psi}{\partial \theta^2} + \cos\theta \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right]$$

Thus $$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

Results at a glance

1. *Cylindrical system* $(u_1, u_2, u_3) = (\rho, \phi, z)$, $h_1 = 1, h_2 = \rho, h_3 = 1$
2. *Spherical system* : $(u_1, u_2, u_3) = (r, \theta, \phi)$, $h_1 = 1, h_2 = r, h_3 = r\sin\theta$
3. $\text{grad}\,\psi = \nabla \psi = \frac{1}{h_1} \Sigma \frac{\partial \psi}{\partial u_1} \hat{e}_1$

4. $\operatorname{div} \vec{A} = \nabla \cdot \vec{A} = \dfrac{1}{h_1 h_2 h_3} \Sigma \dfrac{\partial}{\partial u_1} (A_1 h_2 h_3)$

where $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$

5. $\operatorname{curl} \vec{A} = \nabla \times \vec{A} = \dfrac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$

6. $\nabla^2 \psi = \dfrac{1}{h_1 h_2 h_3} \Sigma \dfrac{\partial}{\partial u_1} \left(\dfrac{h_2 h_3}{h_1} \dfrac{\partial \psi}{\partial u_1} \right)$

Remark : $(u_1, u_2, u_3) = (x, y, z)$ *and* $h_1 = h_2 = h_3 = 1$ *for the cartesian system Results (3) to (6) reduces to the already known definitions in the cartesian system.*

Unit - V

INTEGRAL CALCULUS

5.1 Introduction

We are familar with various methods of integration, definite integrals and the associated application of finding the area under a curve.

In this unit we first discuss the topic **Differentiation under the integral sign**, by which we can evaluate certain definite integrals which are either difficult or impossible by known methods of integration.

Later we discuss three **Reduction Formulae** which will be useful in discussing some more **applications of integral calculus : perimeter, surface area and volume** in respect of certain standard curves.

5.2 Differentiation under the integral sign

This topic deals with the technique of evaluating a definite integral of a function of an independent variable along with a parameter by applying well established rule known as a *Leibnitz rule*. It is important to note that the definite integrals are either difficult or impossible to evaluate by various known methods of integration. Further, starting from the value of one definite integral, applying the rule, we can find the value of an other definite integral which is otherwise difficult/impossible to evaluate

5.2.1 Leibnitz rule for differentiation under the integral sign

If $f(x, \alpha)$, α being the parameter and $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ are continuous functions then it is obvious that $\int_a^b f(x, \alpha)\, dx$ is a function of α, be denoted by $\phi(\alpha)$. That is to say that if

$$\phi(\alpha) = \int_a^b f(x, \alpha)\, dx$$

then Leibnitz rule states that

$$\phi'(\alpha) = \frac{d\phi}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha}[f(x, \alpha)]\, dx \quad \text{where } a \text{ and } b \text{ are constants}$$

In other words the rule means that

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial}{\partial\alpha}\,[f(x,\alpha)]\,dx$$

Working procedure for problems

- The given definite integral be denoted by $\phi(\alpha)$, α being the parameter, x is the variable of integration.
- We find $\phi'(\alpha)$ by applying the rule. That is to differentiate the integrand partially with respect to the parameter within the integral sign.
- We integrate between the given limits to obtain a function of α. That is, if $\phi'(\alpha) = F(\alpha)$ (*say*) then $\phi(\alpha) = \int F(\alpha)\,d\alpha + c$, c being the constant of integration.
- We evaluate c by taking a suitable value for α, with the result we obtain the required $\phi(\alpha)$.

WORKED PROBLEMS

1. [illegible] $\int e^{-\alpha x}\frac{\sin x}{x}\,dx$ by differentiating under the integral [illegible]

$\int \frac{\sin x}{x}\,dx$

>> Let $\phi(\alpha) = \int_0^\infty e^{-\alpha x}\,\frac{\sin x}{x}\,dx$ ... (1)

We have by Leibnitz rule,

$$\phi'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}\left[e^{-\alpha x}\cdot\frac{\sin x}{x}\right] = \int_0^\infty e^{-\alpha x}\cdot(-x)\,\frac{\sin x}{x}\,dx$$

Using $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx)$ where $a = -\alpha,\ b = 1,$

$$\phi'(\alpha) = -\left[\frac{e^{-\alpha x}}{\alpha^2+1}\,(-\alpha\sin x - \cos x)\right]_{x=0}^{\infty}$$

$$= \frac{1}{\alpha^2+1}\left[e^{-\alpha x}\,(\alpha\sin x + \cos x)\right]_{x=0}^{\infty}$$

In other words the rule means that

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial}{\partial\alpha}\,[f(x,\alpha)]\,dx$$

Working procedure for problems

- The given definite integral be denoted by $\phi(\alpha)$, α being the parameter, x is the variable of integration.
- We find $\phi'(\alpha)$ by applying the rule. That is to differentiate the integrand partially with respect to the parameter within the integral sign.
- We integrate between the given limits to obtain a function of α. That is, if $\phi'(\alpha) = F(\alpha)$ (*say*) then $\phi(\alpha) = \int F(\alpha)\,d\alpha + c$, c being the constant of integration.
- We evaluate c by taking a suitable value for α, with the result we obtain the required $\phi(\alpha)$.

WORKED PROBLEMS

[illegible]

>> Let $\phi(\alpha) = \int_0^\infty e^{-\alpha x}\,\dfrac{\sin x}{x}\,dx$ ...(1)

We have by Leibnitz rule,

$$\phi'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}\left[e^{-\alpha x}\cdot\frac{\sin x}{x}\right] = \int_0^\infty e^{-\alpha x}\,(-x)\,\frac{\sin x}{x}\,dx$$

Using $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx)$ where $a = -\alpha$, $b = 1$,

$$\phi'(\alpha) = -\left[\frac{e^{-\alpha x}}{\alpha^2+1}\,(-\alpha\sin x - \cos x)\right]_{x=0}^{\infty}$$

$$= \frac{1}{\alpha^2+1}\left[e^{-\alpha x}\,(\alpha\sin x + \cos x)\right]_{x=0}^{\infty}$$

i.e., $\quad \phi'(\alpha) = \dfrac{1}{\alpha^2+1}\{0 - e^0(\alpha\sin 0 + \cos 0)\} \quad \because \; e^{-\alpha x} \to 0$ as $x \to \infty$.

i.e., $\quad \phi'(\alpha) = \dfrac{-1}{\alpha^2+1}$

$$\phi(\alpha) = -\int \frac{1}{\alpha^2+1}\,d\alpha + c = -\tan^{-1}\alpha + c$$

Thus $\quad \phi(\alpha) = -\tan^{-1}\alpha + c \qquad \ldots (2)$

Now to find c let us put $\alpha = \infty$ in (2) so that we have

$$\phi(\infty) = -\tan^{-1}(\infty) + c$$

Using (1) in L.H.S we have,

$$\int_0^\infty 0 \cdot \frac{\sin x}{x}\,dx = -\frac{\pi}{2} + c \qquad \because \; e^{-\alpha x} \to 0 \text{ as } \alpha \to \infty$$

i.e., $\quad 0 = -\pi/2 + c \qquad \therefore \; c = \pi/2$

Substituting in (2) the required $\phi(\alpha) = -\tan^{-1}\alpha + (\pi/2)$

Thus $\quad \displaystyle\int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx = \frac{\pi}{2} - \tan^{-1}\alpha = \cot^{-1}\alpha$

Now putting $\alpha = 0$, $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} - \tan^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Thus $\quad \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

2. [illegible] $\displaystyle\int \frac{x^{\alpha}-1}{\log} $ [illegible]

the parameter. Hence find $\displaystyle\int \frac{x^3-1}{\log x}\,dx$

>> Let $\quad \phi(\alpha) = \displaystyle\int_0^1 \frac{x^{\alpha}-1}{\log x}\,dx \qquad \ldots (1)$

$$\phi'(\alpha) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{x^{\alpha}-1}{\log x}\right) dx, \text{ by Leibnitz rule}$$

$$= \int_0^1 \frac{1}{\log x}\, x^{\alpha} \log x \, dx \qquad \because \quad \frac{d}{dx}(a^x) = a^x \log a$$

i.e., $$\phi'(\alpha) = \int_0^1 x^{\alpha}\, dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

We have $\phi'(\alpha) = \dfrac{1}{\alpha+1}$

$$\phi(\alpha) = \int \frac{1}{\alpha+1}\, d\alpha + c$$

i.e., $$\phi(\alpha) = \log(\alpha+1) + c \qquad \ldots (2)$$

Putting $\alpha = 0$, we have $\phi(0) = \log 1 + c$

But $\phi(0) = 0$ from (1) and $\log 1 = 0 \quad \therefore \quad c = 0$

Hence we have from (2), $\phi(\alpha) = \log(\alpha+1)$

Thus $$\int_0^1 \frac{x^{\alpha}-1}{\log x}\, dx = \log(\alpha+1)$$

Now putting $\alpha = 3$ we get

$$\int_0^1 \frac{x^3-1}{\log x}\, dx = \log 4 \quad \text{or} \quad \int_0^1 \frac{x^3-1}{\log x}\, dx = \log 2^2 = 2\log 2$$

3. *Evaluate* $\displaystyle\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$ *by differentiating under the integral sign and hence find*

$$\int \frac{\tan^{-1} x}{x(1+x^2)}\, dx$$

>> Let $$\phi(a) = \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\, dx \qquad \ldots (1)$$

$$\phi'(a) = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\tan^{-1} ax}{x(1+x^2)}\right] dx, \text{ by Leibnitz rule.}$$

$$\therefore \quad \phi'(a) = \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{x}{1+(ax)^2}\, dx$$

$$\therefore \quad \phi'(a) = \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)} \qquad \ldots (2)$$

We shall integrate R.H.S by resolving into partial fractions taking $x^2 = t$, only for convenience

Let $\quad \dfrac{1}{(1+t)(1+a^2t)} = \dfrac{A}{1+t} + \dfrac{B}{1+a^2t}$

or $\quad 1 = A(1+a^2t) + B(1+t)$

By putting $t = -1$ $\qquad$ we get $\quad A = \dfrac{1}{1-a^2}$

Also by putting $t = -1/a^2$ $\qquad$ we get $\quad B = \dfrac{-a^2}{1-a^2}$

Hence $\dfrac{1}{(1+t)(1+a^2t)} = \dfrac{1}{1-a^2} \cdot \dfrac{1}{1+t} - \dfrac{a^2}{1-a^2} \cdot \dfrac{1}{1+a^2t}$

Replacing back $t = x^2$ and integrating $w.r.t\ x$ betwee 0 and ∞ we have,

$$\int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)} = \frac{1}{1-a^2} \int_0^\infty \frac{dx}{1+x^2} - \frac{a^2}{1-a^2} \int_0^\infty \frac{dx}{1+(ax)^2}$$

$$i.e. \quad \phi'(a) = \frac{1}{1-a^2} \left\{ [\tan^{-1} x]_0^\infty - a^2 \cdot \frac{1}{a} [\tan^{-1} ax]_0^\infty \right\}$$

$$i.e. \quad \phi'(a) = \frac{1}{1-a^2} \left\{ \frac{\pi}{2} - a\frac{\pi}{2} \right\} = \frac{\pi(1-a)}{2(1-a^2)} = \frac{\pi}{2(1+a)}$$

$$\therefore \quad \phi(a) = \frac{\pi}{2} \int \frac{da}{1+a} + c$$

$$i.e. \quad \phi(a) = \frac{\pi}{2} \log(1+a) + c \qquad \ldots (3)$$

To evaluate c, let us put $a = 0$ in (3)

Hence $\phi(0) = \dfrac{\pi}{2} \log 1 + c$

But $\quad \phi(0) = 0$ from (1) and $\log 1 = 0 \quad \therefore \quad c = 0$

Hence we have from (3), $\phi(a) = \frac{\pi}{2}\log(1+a)$

Thus $$\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a)$$

Now by putting $a = 3$ we get,

$$\int_0^\infty \frac{\tan^{-1} 3x}{x(1+x^2)}\,dx = \frac{\pi}{2}\log 4 = \frac{\pi}{2}\log 2^2 = \frac{\pi}{2}\cdot 2\log 2$$

Thus $$\int_0^\infty \frac{\tan^{-1} 3x}{x(1+x^2)}\,dx = \pi \log 2$$

4. [illegible]

>> Let $$\phi(\alpha) = \int_0^\infty e^{-x^2}\cos\alpha x\,dx \qquad \ldots (1$$

$\therefore$ $$\phi'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}\left(e^{-x^2}\cos\alpha x\right)dx, \text{ by Leibnitz rule.}$$

i.e., $$\phi'(\alpha) = \int_0^\infty e^{-x^2}(-\sin\alpha x\cdot x)\,dx$$

i.e., $$\phi'(\alpha) = \int_0^\infty \sin\alpha x\,(-x e^{-x^2})\,dx$$

Noting that $\int -x e^{-x^2}dx = \frac{1}{2}e^{-x^2}$, we have on integration by parts,

$$\phi'(\alpha) = \left[\sin\alpha x\cdot\frac{1}{2}e^{-x^2}\right]_{x=0}^{\infty} - \int_0^\infty \frac{1}{2}e^{-x^2}\cdot\cos\alpha x\cdot\alpha\,dx$$

i.e., $$\phi'(\alpha) = 0 - \frac{\alpha}{2}\int_0^\infty e^{-x^2}\cos\alpha x\,dx \quad\text{or}\quad \phi'(\alpha) = -\frac{\alpha}{2}\phi(\alpha) \text{ by using (1)}$$

[Here we adopt a different technique to find $\phi(\alpha)$]

$$\frac{\phi'(\alpha)}{\phi(\alpha)} = -\frac{\alpha}{2} \text{ and on integration we have}$$

i.e., $$\int \frac{\phi'(\alpha)}{\phi(\alpha)}\, d\alpha = -\int \frac{\alpha}{2}\, d\alpha + c$$

i.e., $$\log \phi(\alpha) = -\frac{\alpha^2}{4} + c$$

$\Rightarrow$ $$\phi(\alpha) = e^{-\alpha^2/4 + c} \quad \text{or} \quad \phi(\alpha) = e^{c}\, e^{-\alpha^2/4}$$

Putting $\alpha = 0$ we have,

$$\phi(0) = e^{c} \cdot 1$$

$$\int_0^\infty e^{-x^2} \cdot 1\, dx = e^{c}$$

Note $\int_0^\infty e^{-x^2} dx$ *is to be evaluated using gamma functions and it works out to be $\sqrt{\pi}/2$ which is to be assumed here.*

Hence we have $\sqrt{\pi}/2 = e^{c}$, with the result we write,

$$\phi(\alpha) = \sqrt{\pi}/2 \cdot e^{-\alpha^2/4}$$

Thus $$\int_0^\infty e^{-x^2} \cos \alpha x\, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$$

5. By differentiating under the integral sign, prove that $\int_0^\pi \frac{\log(1 + a\cos x)}{\cos x} dx = \pi \sin^{-1} a$

>> Let $$\phi(a) = \int_0^\pi \frac{\log(1 + a\cos x)}{\cos x}\, dx \qquad \ldots (1)$$

$\therefore$ $$\phi'(a) = \int_0^\pi \frac{1}{\cos x} \cdot \frac{1}{1 + a\cos x} \cdot \cos x\, dx, \text{ by Leibnitz rule.}$$

$$\phi'(a) = \int_0^\pi \frac{dx}{1 + a\cos x}$$

We employ a known method to integrate R.H.S by using the substitution $\tan(x/2) = t$. This will give us,

$$\cos x = \frac{1-t^2}{1+t^2} \text{ and } dx = \frac{2dt}{1+t^2} \; ; \; t \text{ varies from } 0 \text{ to } \infty$$

$$\text{Hence } \phi'(a) = \int_0^{\infty} \frac{2\,dt/1+t^2}{1+a\cdot\dfrac{1-t^2}{1+t^2}} = 2\int_0^{\infty} \frac{dt}{(1+t^2)+a(1-t^2)}$$

i.e.,
$$\phi'(a) = 2\int_0^{\infty} \frac{dt}{(1+a)+(1-a)t^2} = \frac{2}{1-a}\int_0^{\infty} \frac{dt}{t^2+\left(\dfrac{1+a}{1-a}\right)}$$

Denoting $b^2 = \dfrac{1+a}{1-a}$ we have,

$$\phi'(a) = \frac{2}{1-a}\int_0^{\infty} \frac{dt}{t^2+b^2}$$

ie.,
$$= \frac{2}{1-a}\left[\frac{1}{b}\tan^{-1}\left(\frac{t}{b}\right)\right]_{t=0}^{\infty} = \frac{2}{(1-a)b}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

ie.,
$$\phi'(a) = \frac{2}{(1-a)}\,\frac{\sqrt{1-a}}{\sqrt{1+a}}\,\frac{\pi}{2}$$

i.e.,
$$\phi'(a) = \frac{\pi}{\sqrt{1-a^2}} \qquad \left|\; \text{Note: } \int_0^{\pi} \frac{dx}{1+a\cos x} = \frac{\pi}{\sqrt{1-a^2}},\; a<1 \right.$$

$$\phi(a) = \pi\int \frac{da}{\sqrt{1-a^2}} + c$$

i.e.,
$$\phi(a) = \pi\sin^{-1}a + c$$

Putting $a = 0$ we have, $\phi(0) = 0 + c$

But $\phi(0) = 0$ from (1) and we have $c = 0$

Hence we get $\phi(a) = \pi\sin^{-1}a$

Thus we have proved that,

$$\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\,dx = \pi\sin^{-1}a$$

>> Let $\phi(a) = \int_0^{\pi} \log(1 + a\cos x)\, dx$ (1)

$\therefore \quad \phi'(a) = \int_0^{\pi} \frac{\cos x}{1 + a\cos x}\, dx,$ by Leibnitz rule

$$= \frac{1}{a}\int_0^{\pi} \frac{a\cos x}{1 + a\cos x}\, dx = \frac{1}{a}\int_0^{\pi} \frac{(1 + a\cos x - 1)}{1 + a\cos x}\, dx$$

i.e., $\phi'(a) = \frac{1}{a}\int_0^{\pi} dx - \frac{1}{a}\int_0^{\pi} \frac{dx}{1 + a\cos x}$

i.e., $\phi'(a) = \frac{\pi}{a} - \frac{1}{a}\cdot\frac{\pi}{\sqrt{1 - a^2}}$ (*Refer previous example*)

$$\phi(a) = \pi\left[\int \frac{da}{a} - \int \frac{da}{a\sqrt{1 - a^2}}\right] + c$$

Using the formula,

$$\int \frac{dx}{x\sqrt{1 - x^2}} = -\operatorname{sech}^{-1} x = -\log\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) + c$$

$$\phi(a) = \pi\left[\log a + \log\left(\frac{1 + \sqrt{1 - a^2}}{a}\right)\right]$$

or $\quad \phi(a) = \pi\log(1 + \sqrt{1 - a^2}) + c$

Putting $a = 0$ we have, $\phi(0) = \pi\log 2 + c$

But $\quad \phi(0) = 0$ from (1) and we have $c = -\pi\log 2$

Hence we get $\phi(a) = \pi\log(1 + \sqrt{1 - a^2}) - \pi\log 2$

Thus we have $\int_0^{\pi} \log(1 + a\cos x)\, dx = \pi\log\left[\frac{1 + \sqrt{1 - a^2}}{2}\right], \quad a < 1$

7. *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx$ *by differentiating under the integral sign.*

>> Let $$\phi(y) = \int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx \qquad \ldots(1)$$

$$\therefore \quad \phi'(y) = \int_0^{\pi/2} \frac{1}{\sin^2 x}\cdot\frac{1}{1+y\sin^2 x}\cdot\sin^2 x\,dx, \text{ by Leibnitz rule}$$

i.e., $$\phi'(y) = \int_0^{\pi/2} \frac{dx}{1+y\sin^2 x}$$

(*We employ a known method to integrate R.H.S*)

$$\phi'(y) = \int_0^{\pi/2} \frac{dx}{\cos^2 x + (1+y)\sin^2 x} = \int_0^{\pi/2} \frac{dx}{\cos^2 x\,[1+(1+y)\tan^2 x]}$$

i.e., $$\phi'(y) = \int_0^{\pi/2} \frac{\sec^2 x\,dx}{1+(1+y)\tan^2 x}$$

Taking $t = \sqrt{1+y}\tan x$ we get $dt = \sqrt{1+y}\sec^2 x\,dx$

Also t varis from 0 to ∞.

Hence $$\phi'(y) = \int_0^{\infty} \frac{dt/\sqrt{1+y}}{1+t^2}$$

i.e., $$\phi'(y) = \frac{1}{\sqrt{1+y}}\left[\tan^{-1} t\right]_0^{\infty} = \frac{1}{\sqrt{1+y}}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

i.e., $$\phi'(y) = \frac{\pi}{2\sqrt{1+y}}$$

$$\therefore \quad \phi(y) = \pi\int\frac{dy}{2\sqrt{1+y}} + c$$

i.e., $$\phi(y) = \pi\sqrt{1+y} + c$$

Putting $y = 0$ we have $\phi(0) = \pi + c$

But $\phi(0) = 0$ from (1) and hence $c = -\pi$

Hence we get $\phi(y) = \pi\sqrt{1+y} - \pi$

Thus we have $$\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx = \pi\left[\sqrt{1+y} - 1\right]$$

$\int \frac{dx}{\alpha - \cos x} = \frac{\pi}{[illegible]}$, $\alpha > 1$, deduce $\int$ [illegible] and also [illegible]

>> Let $\phi(\alpha) = \int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}$

Differentiate *w.r.t.* α

Now $\phi'(\alpha) = \int_0^{\pi} \frac{\partial}{\partial \alpha} \left[\frac{1}{\alpha - \cos x} \right] dx = \pi \frac{d}{d\alpha} (\alpha^2 - 1)^{-1/2}$

where Leibnitz rule is employed for differentiating under the integral sign

i.e., $\int_0^{\pi} \frac{-1}{(\alpha - \cos x)^2}\, dx = \pi \cdot -\frac{1}{2} (\alpha^2 - 1)^{-3/2} \cdot 2\alpha$

or $\int_0^{\pi} \frac{dx}{(\alpha - \cos x)^2} = \frac{\pi\alpha}{(\alpha^2 - 1)^{3/2}}$

Now by putting $\alpha = 2$ we get,

$$\int_0^{\pi} \frac{dx}{(2 - \cos x)^2} = \frac{2\pi}{3\sqrt{3}}$$

Remark *Here it may be observed that starting from the value of one integral, which infact can be obtained comfortably, we have found the value of an other integral easily (which is otherwise difficult to evaluate) by applying Leibnitz rule.*

>> Let $\phi(m) = \int_0^1 x^m\, dx = \frac{1}{m+1}$.

Differentiate w.r.t m, where We shall apply Leibnitz rule to differentiate under the integral sign.

$$\therefore \qquad \int_0^1 \frac{\partial}{\partial m}(x^m)\,dx = \frac{d}{dm}\left(\frac{1}{m+1}\right)$$

$$i.e., \qquad \int_0^1 x^m(\log x)\,dx = \frac{-1}{(m+1)^2}$$

Applying the rule again we have,

$$\int_0^1 x^m \log x\ (\log x)\,dx \quad (-1)(\ \ 2)(m+1)^{\ \ 3} - (-1)^2\, 2!\,(m+1)^{-3}$$

$$i.e., \qquad \int_0^1 x^m(\log x)^2\,dx = (-1)^2\ 2!\,(m+1)^{-3}$$

Applying the rule once again we have

$$\int_0^1 x^m \log x\,(\log x)^2\,dx = (-1)^2\ 2!\,(-3)\,(m+1)^{-4}$$

$$i.e., \qquad \int_0^1 x^m(\log x)^3\,dx = (-1)^3\ 3!\,(m+1)^{-4}$$

Continuing like this, by differentiating n times we get

$$\int_0^1 x^m(\log x)^n\,dx = (-1)^n\, n!\,(m+1)^{-(n+1)}$$

$$\text{Thus} \qquad \int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$$

10. Differentiating under the integral sign the integral $\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\pi/a}$ show that

>> Let $\phi(a) = \int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\pi/a} = \frac{\sqrt{\pi}}{2}\,a^{-1/2}$

$$\phi'(a) = \int_0^\infty e^{-ax^2}(-x^2)\,dx = \frac{\sqrt{\pi}}{2}\cdot -\frac{1}{2}a^{-3/2}$$

where we have employed Leibnitz rule to differentiate under the integral sign

Differentiating again *w.r.t* a we have,

$$\phi''(a) = \int_0^\infty e^{-ax^2}(-x^2)^2\,dx = \frac{\sqrt{\pi}}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\,a^{-5/2}$$

Similarly we have,

$$\phi'''(a) = \int_0^\infty e^{-ax^2}(-x^2)^3\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\,a^{-7/2}$$

Continuing like this, on differentiating n times we have,

$$\phi^{(n)}(a) = \int_0^\infty e^{-ax^2}(-x^2)^n\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdots\frac{-(2n-1)}{2}\,a^{-(2n+1)/2}$$

or $$\int_0^\infty e^{-ax^2}(x^2)^n(-1)^n\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}(-1)^n a^{-n-\frac{1}{2}}$$

Thus $$\int_0^\infty e^{-ax^2}x^{2n}\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\mathbf{1\cdot 3\cdot 5\cdots(2n-1)}}{2^n a^{n+\frac{1}{2}}}$$

EXERCISES

Evaluate the following integrals by differentiating under the integral sign [1 to 3]

1. $\displaystyle\int_0^{\infty} \frac{e^{-x}}{x}(1-e^{-ax})\,dx, \quad a > -1$

2. $\displaystyle\int_0^{\infty} x\, e^{-x^2} \sin ax\, dx$

3. $\displaystyle\int_0^{\pi} \frac{\log(1+\sin a \cos x)}{\cos x}\,dx$

4. Differentiating $\displaystyle\int_0^{x} \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ under the integral sign, evaluate

$$\int_0^{x} \frac{dx}{(x^2+a^2)^2}$$

5. Given that $\displaystyle\int_0^{\infty} \frac{dx}{x^2+a} = \frac{\pi}{2\sqrt{a}}, a > 0$ show that

$$\int_0^{\infty} \frac{dx}{(x^2+a)^{n+1}} = \frac{\pi}{2} \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!\, a^{n+\frac{1}{2}}}$$

6. If $\phi(y) = \displaystyle\int_0^{\infty} e^{-x^2} \sin 2yx\, dx$, show that $\dfrac{d\phi(y)}{dy} + 2y\,\phi(y) = 1$

ANSWERS

1. $\log(1+a)$

2. $\dfrac{\sqrt{\pi}\, a}{4}\, e^{-a^2/4}$

3. πa

4. $\dfrac{x}{2a^2(x^2+a^2)} + \dfrac{1}{2a^3}\tan^{-1}(x/a)$

Reduction Formulae

Reduction formulae is basically a recurrence relation which reduces integral of functions of higher degree in the form $\int [f(x)]^n\, dx$, $\int [f(x)]^m [g(x)]^n\, dx$ (*where m and n are non negative integers*) to lower degree. The successive application of the recurrence relation finally end up with a function of degree 0 or 1 so that we can easily complete the integration process.

We discuss three standard reduction formulae in the form of indefinite integrals and the evaluation of them with standard limits of integration.

Let $I_n = \int \sin^n x \, dx$

$$\int \sin^{n-1} x \cdot \sin x \, dx = \int u\,v \, dx \text{ (say)}$$

We have the rule of integration by parts,

$$\int uv \, dx = u \int v \, dx - \int \int v \, dx \cdot u' \, dx$$

$$I_n = \sin^{n-1} x \, (-\cos x) - \int (-\cos x)\,(n-1)\,\sin^{n-2} x \cdot \cos x \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \,(1-\sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

ie., $I_n = -\sin^{n-1} x \cdot \cos x + (n-1)\, I_{n-2} - (n-1)\, I_n$

ie., $I_n\,[1+(n-1)] = -\sin^{n-1} x \cdot \cos x + (n-1)\, I_{n-2}$

$$I_n = \int \sin^n x \, dx = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} I_{n-2} \qquad (1)$$

This is the required reduction formula

) *To find* $\int \sin^4 x \, dx$

> Comparing with the L.H.S. of (1), we need to take $n = 4$ and use the established sult

$$I_4 = \int \sin^4 x \, dx = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} I_2$$

e need to apply the result (1) again by taking $n = 2$

$$I_4 = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left\{\frac{-\sin x \cos x}{2} + \frac{1}{2} I_0\right\}$$

e cannot find I_0 from (1). But basically we have

$$I_0 = \int \sin^0 x \, dx = \int 1 \, dx = x$$

Thus $\quad I_4 = \int \sin^4 x \ dx = \dfrac{-\sin^3 x \ \cos x}{4} - \dfrac{3}{8} \sin x \ \cos x + \dfrac{3x}{8} + c$

(ii) *To find* $\int \sin^5 x \ dx$

>> $\quad I_5 = \int \sin^5 x \ dx = \dfrac{-\sin^4 x \ \cos x}{5} + \dfrac{4}{5} I_3$

ie., $\quad = \dfrac{-\sin^4 x \ \cos x}{5} + \dfrac{4}{5}\left\{\dfrac{-\sin^2 x \ \cos x}{3} + \dfrac{2}{3} I_1\right\}$

But $\quad I_1 = \int \sin^1 x \ dx = \int \sin x \ dx = -\cos x$

$$\int \sin^5 x \ dx = \frac{-\sin^4 x \ \cos x}{5} - \frac{4 \sin^2 x \ \cos x}{15} - \frac{8}{15} \cos x + c$$

Corollary : Evaluation of $\displaystyle\int_0^{\pi/2} \sin^n x \ dx$

$$\text{Let } I_n = \int_0^{\pi/2} \sin^n x \ dx$$

Equation (1) *must be established first*

$$\text{from (1) } I_n = -\left[\frac{\sin^{n-1} x \ \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

But $\quad \cos(\pi/2) = 0 = \sin 0.$

Thus $\quad I_n = \dfrac{n-1}{n} I_{n-2}$ $\quad$ . . (2)

We use this recurrence relation to find I_{n-2} by simply replacing n by $(n-2)$.

ie , $\quad I_{n-2} = \dfrac{n-3}{n-2} I_{n-4}$

Hence $\quad I_n = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} I_{n-4}$, by back substitution.

Similarly from (2) $I_{n-4} = \dfrac{n-5}{n-4} I_{n-6}$

Hence $\quad I_n = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} I_{n-6}$, again by back substitution

Continuing like this, the reduction process will end up as follows

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}\, I_1 \text{ if } n \text{ is odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2}\, I_0 \text{ if } n \text{ is even}$$

But $\quad I_1 = \int_0^{\pi/2} \sin x \; dx = -\left[\cos x\right]_0^{\pi/2} = -(0-1) = 1$

and $\quad I_0 = \int_0^{\pi/2} \sin^0 x \; dx = \int_0^{\pi/2} dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$

hus we have,

$$\int_0^{\pi/2} \sin^n x \; dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

lustrations : *To find* (i) $\int_0^{\pi/2} \sin^4 x \; dx$ (ii) $\int_0^{\pi/2} \sin^5 x \; dx$

$$\int_0^{\pi/2} \sin^4 x \; dx = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$$

) $\quad \int_0^{\pi/2} \sin^5 x \; dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$

$$I_n = \int \cos^n x \; dx$$

$$= \int \cos^{n-1} x \cdot \cos x \; dx$$

egrating by parts we have,

$$I_n = \cos^{n-1} x \cdot \sin x - \int \sin x \; (n-1) \cos^{n-2} x \; (-\sin x) \; dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x \; dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x (1-\cos^2 x)\, dx$$

$$\cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$I_n = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} - (n-1) I_n$$

i.e., $$I_n [1 + (n-1)] = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2}$$

$$I \quad \int \cos^n x \, dx = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n} I_{n-2} \qquad \ldots (1)$$

Next, let $I_n = \int\limits_0^{\pi/2} \cos^n x \, dx$

from (1), $I_n = \left[\dfrac{\cos^{n-1} x \sin x}{n} \right]_0^{\pi/2} + \dfrac{n-1}{n} I_{n-2}$

But $\cos(\pi/2) = 0 = \sin 0$

Thus $I_n = \dfrac{n-1}{n} I_{n-2}$ $\qquad$. (2)

Remark *This result is same as (2) of 5 31 and proceeding on the same lines as in 5 31 we can obtain the result which is identically same as that of* $\int\limits_0^{\pi/2} \sin^n x \, dx$

Infact we can easily conclude that $\int\limits_0^{\pi/2} \sin^n x \, dx = \int\limits_0^{\pi/2} \cos^n x \, dx$ *because of a property of definite integrals,* $\int\limits_0^a f(x)\, dx = \int\limits_0^a f(a-x)\, dx$ *as we have* $f(x) = \sin^n x$,

$f(\pi/2 - x) = [\sin(\pi/2 - x)]^n = (\cos x)^n = \cos^n x$

$$\int\limits_0^{\pi/2} \cos^n x \, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{2}{3} & \text{if } n \text{ is odd} \\[2ex] \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

Illustrations

(i) *To find* $\int \cos^6 x \, dx$

>> We have from (1) by putting $n = 6$,

$$I_6 = \int \cos^6 x \, dx = \frac{\cos^5 x \sin x}{6} + \frac{5}{6} I_4$$

i.e.,
$$= \frac{\cos^5 x \sin x}{6} + \frac{5}{6} \left\{ \frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2 \right\}$$

$$= \frac{\cos^5 x \sin x}{6} + \frac{5}{24} \cos^3 x \sin x + \frac{5}{8} \left\{ \frac{\cos x \sin x}{2} + \frac{1}{2} I_0 \right\}$$

But $\quad I_0 = \int \cos^0 x \, dx = \int 1 \, dx = x$

$$\therefore \quad \int \cos^6 x \, dx = \frac{\cos^5 x \sin x}{6} + \frac{5 \cos^3 x \sin x}{24} + \frac{5 \cos x \sin x}{16} + \frac{5x}{16} + c$$

To find **(ii)** $\displaystyle\int_0^{\pi/2} \cos^6 x \, dx$ **(iii)** $\displaystyle\int_0^{\pi/2} \cos^7 x \, dx$

(ii) $$\int_0^{\pi/2} \cos^6 x \, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$$

(iii) $$\int_0^{\pi/2} \cos^7 x \, dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{35}$$

5.33 Reduction formula for $\int \sin^m x \cos^n x \, dx$ and $\displaystyle\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ where m and n are positive integers.

Let $\quad I_{m,n} = \int \sin^m x \, (\cos^n x \, dx)$

$$= \int \sin^{m-1} x \, (\sin x \cos^n x) \, dx = \int uv \, dx \quad (say)$$

We have $\int uv \, dx = u \int v \, dx - \int \int v \, dx \cdot u' \, dx$

Here $\quad \int v \, dx = \int \sin x \cos^n dx$

Put $\quad \cos x = t \quad \therefore \quad -\sin x \, dx = dt$

Hence $\quad \int v\,dx = \int -t^n\,dt = -\dfrac{t^{n+1}}{n+1} = -\dfrac{\cos^{n+1}x}{n+1}$

Now $\quad I_{m,n} = (\sin^{m-1}x)\left(\dfrac{-\cos^{n+1}x}{n+1}\right) - \int \dfrac{-\cos^{n+1}x}{n+1}\,(m-1)\sin^{m-2}x\cos x\,dx$

ie, $\quad = -\dfrac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2}x\cos^{n+2}x\,dx$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cdot\cos^n x\cdot\cos^2 x\,dx$$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,(1-\sin^2 x)\,dx$$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,dx - \frac{m-1}{n+1}\int \sin^m x\cos^n x\,dx$$

$$I_{m,n} = \frac{-\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}I_{m-2,n} - \frac{m-1}{n+1}I_{m,n}$$

ie, $\quad I_{m,n}\left[1+\dfrac{m-1}{n+1}\right] = \dfrac{1}{n+1}\left[-\sin^{m-1}x\cos^{n+1}x + (m-1)\,I_{m-2,n}\right]$

$$I_{m,n}\left[\frac{m+n}{n+1}\right] = \frac{1}{n+1}\left[-\sin^{m-1}x\cos^{n+1}x + (m-1)\,I_{m-2,n}\right]$$

$\therefore \quad I_{m,n} = \int \sin^m x\cos^n x\,dx = -\dfrac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \dfrac{m-1}{m+n}I_{m-2,n} \qquad \ldots (1)$

Note : *If we decompose* $\sin^m x\cos^n x = (\sin^m x\cos x)\cos^{n-1}x$ *and integrate by parts, by taking* $u = \cos^{n-1}x,\ v = \sin^m x\cos x$ *we can obtain*

$$I_{m,n} = \frac{-\sin^{m+1}x\,\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}I_{m,n-2} \qquad \ldots (2)$$

Now, let $I_{m,n} = \displaystyle\int_0^{\pi/2} \sin^m x\cos^n x\,dx$

$\therefore \quad$ from (1), $I_{m,n} = -\left[\dfrac{\sin^{m-1}x\cos^{n+1}x}{m+n}\right]_0^{\pi/2} + \dfrac{m-1}{m+n}I_{m-2,n}$

i.e., $\quad I_{m,n} = \dfrac{m-1}{m+n}I_{m-2,n} \quad [\because \cos(\pi/2) = 0 = \sin 0]$

$$\therefore \quad I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}$$

Hence $I_{m,n} = \dfrac{m-1}{m+n} \cdot \dfrac{m-3}{m+n-2} I_{m-4,n}$ by back substitution.

Continuing like this we obtain

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \times \begin{cases} \dfrac{2}{3+n} I_{1,n} & \text{if } m \text{ is odd} \\ \dfrac{1}{2+n} I_{0,n} & \text{if } m \text{ is even} \end{cases}$$

Now $\quad I_{1,n} = \displaystyle\int_0^{\pi/2} \sin x \cos^n x \, dx = -\left[\frac{\cos^{n+1} x}{n+1}\right]_0^{\pi/2} = \frac{1}{n+1}$

and $\quad I_{0,n} = \displaystyle\int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \times \frac{1}{n+1} \quad \text{if } m \text{ is odd and } n \text{ is even or odd.}$$

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n}$$

$$\times \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} & \text{if } m \text{ is even and } n \text{ is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } m \text{ is even and } n \text{ is even} \end{cases}$$

Note: *This reduction formula for all the cases can be represented as follows.*

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)(m+n-4)} \times k$$

where $k = \pi/2$ when m and n are even and $k = 1$ other wise. This is known as ***Walli's rule.***

Illustrations

(i) $\displaystyle\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx = \frac{[(4)(2)][(3)(1)]}{9 \times 7 \times 5 \times 3 \times 1} = \frac{8}{315}$

(ii) $$\int_0^{\pi/2} \sin^7 x \cos^5 x\, dx = \frac{[(6)(4)(2)][(4)(2)]}{12 \times 10 \times 8 \times 6 \times 4 \times 2} = \frac{1}{120}$$

(iii) $$\int_0^{\pi/2} \sin^6 x \cos^5 x\, dx = \frac{[(5)(3)(1)][(4)(2)]}{11 \times 9 \times 7 \times 5 \times 3 \times 1} = \frac{8}{693}$$

(iv) $$\int_0^{\pi/2} \sin^8 x \cos^6 x\, dx = \frac{[(7)(5)(3)(1)][(5)(3)(1)]}{14 \times 12 \times 10 \times 8 \times 6 \times 4 \times 2} \cdot \frac{\pi}{2} = \frac{5\pi}{4096}$$

WORKED PROBLEMS

Evaluation of definite integrals

Note 1. *The pre requisite for the evaluation of definite integrals are its various properties. In the ultimate step, the evaluation is completed with the help of the reduction formulae*

2. *In the cases of definite integrals involving algebraic functions like*

(i) $a - x$, we can use the substitution $x = a \sin^2 \theta$

(ii) $a + x$, we can use the substitution $x = a \tan^2 \theta$

To remember

(i) $$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \times k$$

where $k = \pi/2$ *only when n is even*

(ii) $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots} \times k$$

where $k = \pi/2$ *only when m and n are even and $k = 1$, otherwise.*

Evaluate the following integrals

11. $\int_0^{\pi} \sin^5 (x/2)\, dx$ **12.** $\int_0^{\pi} \sin^4 x\, dx$ **13.** $\int_0^{\pi} x \sin^8 x\, dx$

14. $\int_0^{\pi} x \cos^6 x\, dx$ **15.** $\int_{-\pi/2}^{\pi/2} \cos^8 x\, dx$ **16.** $\int_0^{\pi} x \sin^2 x \cos^4 x\, dx$

17. $\int_0^{\pi} \sin^6 x \cos^4 x\, dx$

11. Let $I = \int_0^{\pi} \sin^5 (x/2)\, dx$

Put $x/2 = y \quad \therefore \quad dx = 2\, dy$ If $x = 0,\ y = 0$; If $x = \pi,\ y = \pi/2$

$$\therefore \quad I = 2 \int_0^{\pi/2} \sin^5 y\, dy$$

$$= 2 \cdot \frac{4}{5} \cdot \frac{2}{3} \text{ by reduction formula.}$$

Thus $I = 16/15$

---

12. Let $I = \int_0^{\pi} \sin^4 x\, dx$

If $f(x) = \sin^4 x$ and $2a = \pi$ or $a = \pi/2$

$$f(2a-x) = \sin^4(\pi - x) = \sin^4 x = f(x) \quad ie.,\ f(2a-x) = f(x)$$

Thus by the property $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$ we have,

$$I = 2 \int_0^{\pi/2} \sin^4 x\, dx = 2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ by reduction formula.}$$

Thus $I = 3\pi/8$

---

13. Let $I = \int_0^{\pi} x \sin^8 x\, dx$

We have the property $\int_0^{a} f(x)\, dx = \int_0^{a} f(a-x)\, dx$

$$\therefore \quad I = \int_0^{\pi} (\pi - x) \sin^8(\pi - x)\, dx = \int_0^{\pi} (\pi - x) \sin^8 x\, dx$$

$$= \pi \int_0^{\pi} \sin^8 x\, dx - \int_0^{\pi} x \sin^8 x\, dx$$

$$I = \pi \int_0^{\pi} \sin^8 x\,dx - I \quad \text{or} \quad 2I = \pi \cdot 2 \int_0^{\pi/2} \sin^8 x\,dx \quad \textit{(As in Example 12)}$$

Hence $I = \pi \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, by reduction formula

Thus $\quad I = 35\pi^2 / 256$

---

14. Let $I = \int_0^{\pi} x \cos^6 x\ dx$

$$I = \int_0^{\pi} (\pi - x)\ \cos^6\ (\pi - x)\ dx = \int_0^{\pi} (\pi - x)\ \cos^6 x\ dx$$

$$I = \pi \int_0^{\pi} \cos^6 x\ dx - \int_0^{\pi} x \cos^6 x\ dx = \pi \int_0^{\pi} \cos^6 x\ dx - I$$

$$2I = \pi \cdot 2 \int_0^{\pi/2} \cos^6 x\ dx$$

$$I = \pi \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{by reduction formula.}$$

Thus $\quad I = 5\pi^2 / 32$

---

15. Let $I = \int_{-\pi/2}^{\pi/2} \cos^8 x\ dx$

We have the property : $\int_{-a}^{a} f(x)\ dx = 2 \int_0^{a} f(x)\ dx$ if $f(-x) = f(x)$

Here $\quad \cos^8 (-x) = \cos^8 x$ and hence we have,

$$I = 2 \int_0^{\pi/2} \cos^8 x\ dx$$

$$= 2 \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{by reduction formula}$$

Thus $\quad I = 35\pi / 128$

16. Let $I = \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

$$I = \int_0^{\pi} (\pi - x) \sin^2 (\pi - x) \cos^4 (\pi - x) \, dx \text{ , by a property.}$$

$$= \int_0^{\pi} (\pi - x) \sin^2 x \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx - \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx - I$$

$$2I = \pi \cdot 2 \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

$$\therefore \quad I = \pi \cdot \frac{(1) \cdot (3)\,(1)}{6 \times 4 \times 2} \cdot \frac{\pi}{2} \quad \text{by reduction formula.}$$

Thus $I = \pi^2 / 32$

---

17. Let $I = \int_0^{\pi} \sin^6 x \cos^4 x$

$$I = 2 \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx \text{ , by a property.}$$

$$I = 2 \cdot \frac{[(5)(3)(1)]\,[(3)(1)]}{10 \times 8 \times 6 \times 4 \times 2} \cdot \frac{\pi}{2} \quad \text{by reduction formula.}$$

Thus $I = 3\pi / 256$

---

18. *Evaluate* $\int \cos^4 3x \sin^2 6x \, dx$ *using reduction formula*

>> Let $I = \int_0^{\pi/6} \cos^4 3x \sin^2 6x \, dx$

$\sin 6x = 2\sin 3x \cos 3x \quad \because \quad \sin\theta = 2\sin(\theta/2)\cos(\theta/2)$

$$\therefore \qquad I = \int_0^{\pi/6} \cos^4 3x\ (2\sin 3x \cos 3x)^2\, dx$$

ie., $$I = 4\int_0^{\pi/6} \sin^2 3x \cos^6 3x\, dx$$

Put $3x = y \quad \therefore \quad dx = dy/3.$

If $x = 0,\ y = 0 \quad ; \quad$ If $x = \pi/6,\ y = \pi/2$

$$\therefore \qquad I = 4\int_{y=0}^{\pi/2} \sin^2 y \cos^6 y\ \frac{dy}{3} = \frac{4}{3}\int_0^{\pi/2} \sin^2 y \cos^6 y\, dy$$

$$I = \frac{4}{3}\left[\frac{(1)\cdot(5)(3)(1)}{8\times 6\times 4\times 2}\cdot\frac{\pi}{2}\right] \text{ by reduction formula.}$$

Thus $I = 5\pi/192$

---

19. *Evaluate* $\displaystyle\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\, d\theta$

>> Let $$I = \int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\, d\theta$$

ie., $$I = \int_0^{\pi} \frac{[2\sin(\theta/2)\cos(\theta/2)]^4\, d\theta}{[2\cos^2(\theta/2)]^2}$$

$$= \int_0^{\pi} \frac{16\sin^4(\theta/2)\cos^4(\theta/2)}{4\cos^4(\theta/2)}\, d\theta$$

$$I = 4\int_0^{\pi} \sin^4(\theta/2)\, d\theta$$

Put $\theta/2 = \phi \quad \therefore \quad d\theta = 2\, d\phi$ and ϕ varies from 0 to $\pi/2$.

$$I = 4 \int_{\phi=0}^{\pi/2} \sin^4 \phi \cdot 2d\phi$$

ie., $I = 8 \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}$, by reduction formula.

Thus $I = 3\pi / 2$

20. Evaluate $\int \frac{x^4}{\sqrt{4-x^2}} \, dx$

>> Let $I = \displaystyle\int_0^2 \frac{x^4}{\sqrt{4-x^2}} \, dx$

Put $x^2 = 4\sin^2\theta$ or $x = 2\sin\theta$ $\therefore$ $dx = 2\cos\theta \, d\theta$,

θ varies from 0 to $\pi/2$ and $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$

$$I = \int_{\theta=0}^{\pi/2} \frac{16\sin^4\theta \cdot 2\cos\theta \, d\theta}{2\cos\theta} = 16 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

Hence $I = 16 \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}$, by reduction formula.

Thus $I = 3\pi$

>> Let $I = \displaystyle\int_0^1 x^2 (1-x^2)^{3/2} \, dx$

Put $x = \sin\theta$ $\therefore$ $dx = \cos\theta \, d\theta$ and θ varies from 0 to $\pi/2$.

$$(1-x^2)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$$

$$I = \int_0^{\pi/2} \sin^2\theta \cos^3\theta \cos\theta \, d\theta = \int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$$

Hence $I = \dfrac{(1)\cdot(3)(1)}{6\times4\times2}\cdot\dfrac{\pi}{2}$ by reduction formula.

Thus $I = \pi/32$

22. *Evaluate* $\int\limits_0 x^{3/2}(1-x)^{3/2}$

>> Let $I = \int\limits_0^1 x^{3/2}(1-x)^{3/2}\,dx$

Put $x = \sin^2\theta,\ dx = 2\sin\theta\cos\theta\,d\theta$ and θ varies from 0 to $\pi/2$

Also $(1-x)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$

$$\therefore \quad I = \int\limits_{\theta=0}^{\pi/2}\sin^3\theta\cos^3\theta\cdot2\sin\theta\cos\theta\,d\theta$$

i.e., $$I = 2\int\limits_0^{\pi/2}\sin^4\theta\cos^4\theta\,d\theta$$

Hence $I = 2\cdot\dfrac{[(3)(1)][(3)(1)]}{8\times6\times4\times2}\cdot\dfrac{\pi}{2}$ by reduction formula.

Thus $I = 3\pi/128$

23. [illegible]

>> Let $I = \int\limits_0^1 \sin^{-1}x\cdot x^2\,dx$

Integrating by parts we get,

$$I = \left[\sin^{-1}x\cdot\frac{x^3}{3}\right]_0^1 - \int\limits_0^1\frac{x^3}{3}\,\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \left(\frac{\pi}{2}\cdot\frac{1}{3} - 0\right) - \frac{1}{3}\int\limits_0^1\frac{x^3}{\sqrt{1-x^2}}\,dx$$

Put $x = \sin\theta \quad \therefore\ dx = \cos\theta\,d\theta$ and θ varies from 0 to $\pi/2$

$$I = \frac{\pi}{6} - \frac{1}{3} \int_{\theta=0}^{\pi/2} \frac{\sin^3\theta \cdot \cos\theta \, d\theta}{\cos\theta}$$

$$I = \frac{\pi}{6} - \frac{1}{3} \int_{0}^{\pi/2} \sin^3\theta \, d\theta$$

$$I = \frac{\pi}{6} - \frac{1}{3} \cdot \frac{2}{3} \quad \text{by applying reduction formula.}$$

Thus $\quad \boldsymbol{I = \frac{1}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)}$

24. *Evaluate the* $\int x^2 \sqrt{2ax - x^2} \, dx$

>> Let $I_1 = \int_0^{2a} x^2 \sqrt{2ax - x^2} \, dx$

Put $\quad x = 2a \sin^2\theta \quad \therefore \quad dx = 4a \sin\theta \cos\theta \, d\theta, \quad \theta$ varies from 0 to $\pi/2$

Also $\quad \sqrt{2ax - x^2} = \sqrt{4a^2 \sin^2\theta - 4a^2 \sin^4\theta}$

i.e., $\quad = \sqrt{4a^2 \sin^2\theta \, (1 - \sin^2\theta)} = \sqrt{4a^2 \sin^2\theta \cos^2\theta} = 2a \sin\theta \cos\theta$

$$\therefore \quad I_1 = \int_{\theta=0}^{\pi/2} 4a^2 \sin^4\theta \cdot 2a \sin\theta \cos\theta \cdot 4a \sin\theta \cos\theta \, d\theta$$

$$= 32a^4 \int_0^{\pi/2} \sin^6\theta \cos^2\theta \, d\theta$$

$$= 32a^4 \, \frac{[(5)(3)(1)][(1)]}{8 \times 6 \times 4 \times 2} \cdot \frac{\pi}{2} \quad \text{by reduction formula}$$

Thus $\quad \boldsymbol{I_1 = 5\pi a^4 / 8}$

(ii) $$I_2 = \int_0^{\pi/2} \frac{4a^2 \sin^4\theta}{2a\sin\theta\cos\theta} \cdot 4a\sin\theta\cos\theta \; d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 8a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{by reduction formula}$$

Thus $\quad \boldsymbol{I_2 = 3\pi a^2 / 2}$

25. *Evaluate* $\int x\sqrt{a}$

>> Let $I = \int_0^a x\sqrt{ax - x^2}$

Put $\quad x = a\sin^2\theta \quad \therefore \quad dx = 2a\sin\theta\cos\theta \, d\theta, \quad \theta$ varies from 0 to $\pi/2$

Also $\quad \sqrt{ax - x^2} = \sqrt{a^2\sin^2\theta - a^2\sin^4\theta} = \sqrt{a^2\sin^2\theta\,(1 - \sin^2\theta)}$

$$= \sqrt{a^2\sin^2\theta\cos^2\theta} = a\sin\theta\cos\theta$$

$\therefore$ $$I = \int_0^{\pi/2} a\sin^2\theta \cdot a\sin\theta\cos\theta \cdot 2a\sin\theta\cos\theta \, d\theta$$

$$= 2a^3 \int_0^{\pi/2} \sin^4\theta\cos^2\theta \, d\theta$$

$$= 2a^3 \cdot \frac{(3\cdot 1)(1)}{6\times 4\times 2} \cdot \frac{\pi}{2} \quad \text{by reduction formula.}$$

Thus $\quad \boldsymbol{I = \pi a^3 / 16}$

26. *Show that when n is a positive integer* $\int_0^{2a} x^n \sqrt{2ax - x^2}\, dx = \pi a^2 \left(\frac{a}{2}\right)^n \frac{(2n+1)!}{(n+2)!\, n!}$

and hence deduce the value of $\int_0^{2a} x^3 \sqrt{2ax - x^2}\, dx$

>> Let $\displaystyle I = \int_0^{2a} x^n \sqrt{2ax - x^2}\, dx$

Put $x = 2a \sin^2\theta \quad \therefore \quad dx = 4a \sin\theta \cos\theta \, d\theta$

θ varies from 0 to $\pi/2$. Also $\sqrt{2ax - x^2} = 2a \sin\theta \cos\theta$

$$I = \int_{\theta=0}^{\pi/2} (2a)^n \sin^{2n}\theta \cdot 2a \sin\theta\cos\theta \cdot 4a \sin\theta \cos\theta \, d\theta$$

$$= 2^{n+3} a^{n+2} \int_0^{\pi/2} \sin^{2n+2}\theta \cos^2\theta \, d\theta$$

Here $(2n+2)$ is an even integer and hence the application of the reduction formula will give us

$$I = 2^{n+3}\, a^{n+2}\, \frac{[(2n+1)(2n-1)(2n-3)\cdots 1]\cdot 1}{(2n+4)(2n+2)(2n)\cdots 2} \cdot \frac{\pi}{2}$$

$$= 2^{n+2}\, a^{n+2}\, \frac{(2n+1)(2n-1)(2n-3)\cdots 1 \cdot \pi}{2(n+2)\, 2(n+1)\, 2n\, 2(n-1)\cdots 2\cdot 1}$$

$$= 2^{n+2}\, a^{n+2} \cdot \frac{(2n+1)(2n-1)(2n-3)\cdots 1\cdot\pi}{2^{n+2}(n+2)!}$$

Multiplying both the numerator and the denominator by $2n(2n-2)(2n-4)\cdots 2$ in order to obtain $(2n+1)!$ in the numerator we have,

$$I = a^{n+2} \cdot \frac{(2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots 2\,1\;\pi}{(n+2)!\; 2n\,(2n-2)(2n-4)\cdots 2}$$

$$= \frac{a^{n+2}\cdot(2n+1)!\;\pi}{(n+2)!\; 2n\cdot 2(n-1)\cdot 2(n-2)\cdots 2.1}$$

$$= \frac{a^{n+2}\cdot(2n+1)!\;\pi}{(n+2)!\; 2^n\, n!}$$

Thus $\displaystyle I = \pi a^2 \left(\frac{a}{2}\right)^n \frac{(2n+1)!}{(n+2)!\,n!}$ as required.

Now putting $n = 3$ we get,

$$I = \int_0^{2a} x^3 \sqrt{2ax - x^2}\, dx = \pi a^2 \left(\frac{a}{2}\right)^3 \frac{7!}{5!\,3!}$$ by the above result.

Thus $\quad I = \dfrac{\pi a^5}{8} \cdot \dfrac{7 \cdot 6\,(5!)}{5! \cdot 3!} = \dfrac{7\pi a^5}{8}$

27. *Using reduction formula evaluate* [illegible]

>> Let $\quad I = \displaystyle\int_0^1 \frac{x^9}{\sqrt{1-x^2}}\, dx$

Put $\quad x = \sin\theta \quad \therefore\ dx = \cos\theta\, d\theta$ and θ varies from 0 to $\pi/2$.

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\sin^9\theta}{\cos\theta} \cos\theta\, d\theta = \int_0^{\pi/2} \sin^9\theta\, d\theta$$

Hence $\; I = \dfrac{8}{9} \cdot \dfrac{6}{7} \cdot \dfrac{4}{5} \cdot \dfrac{2}{3}$ by reduction formula.

Thus $\quad I = 128/315$

28. [illegible]

>> Let $\; I = \displaystyle\int_0^{\infty} \frac{x^4}{(1+x^2)^4}\, dx$

Put $\quad x = \tan\theta \quad \therefore \quad dx = \sec^2\theta\, d\theta$

If $\quad x = 0,\ \theta = 0 \quad ; \quad$ If $\quad x = \infty,\ \theta = \pi/2$

Also $\quad (1+x^2)^4 = (1+\tan^2\theta)^4 = (\sec^2\theta)^4 = \sec^8\theta$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\tan^4\theta}{\sec^8\theta} \sec^2\theta\, d\theta = \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^6\theta}\, d\theta$$

$$= \int_0^{\pi/2} \cos^6\theta \, \frac{\sin^4\theta}{\cos^4\theta}\, d\theta = \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\,\theta$$

Hence $I = \dfrac{(3)(1)\cdot(1)}{6\times4\times2}\cdot\dfrac{\pi}{2}$ by reduction formula

Thus $\quad I = \pi / 32$

29. *Evaluate* $\int \dfrac{x}{(1+x^2)^{7/2}}\, dx$

>> Put $x = \tan\theta \quad \therefore \quad dx = \sec^2\theta \, d\,\theta$ and θ varies from 0 to $\pi/2$

Also $(1+x^2)^{7/2} = (\sec^2\theta)^{7/2} = \sec^7\theta$

$$I = \int_{\theta=0}^{\pi/2} \frac{\tan^2\theta\cdot\sec^2\theta}{\sec^7\theta}\, d\theta = \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^5\theta}\, d\,\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\, \cos^3\theta \, d\theta$$

Hence $I = \dfrac{(1)\cdot(2)}{5\times3\times1} = \dfrac{2}{15}$ by reduction formula.

Thus $\quad I = 2/15$

30. *Evaluate* [illegible]

>> Let $\quad I = \displaystyle\int_0^{\infty} \frac{x^2}{(1+x^6)^{7/2}}\, dx$

Put $\quad x^6 = \tan^2\theta$ *ie* $x = (\tan^2\theta)^{1/6} = \tan^{1/3}\theta$

$dx = \dfrac{1}{3}\tan^{-2/3}\theta\, \sec^2\theta \, d\theta$ and θ varies from 0 to $\pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\tan^{2/3}\theta}{\sec^7\theta}\;\frac{1}{3}\tan^{-2/3}\theta\, \sec^2\theta\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^5\theta \, d\,\theta$$

$$\frac{1}{3} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{45} \text{ by reduction formula.}$$

Thus $\quad I = 8/45$

EXERCISES

Evaluate the following .

1. $\displaystyle\int_0^{\pi} x \sin^5 x \; dx$

2. $\displaystyle\int_0^{\pi/6} \sin^5 3x \; dx$

3. $\displaystyle\int_0^{\pi/4} \cos^6 2x \; dx$

4. $\displaystyle\int_0^{\pi} x \sin^6 x \; \cos^4 x \, dx$

5. $\displaystyle\int_0^{2\pi} \sin^4 x \; \cos^6 x \; dx$

6. $\displaystyle\int_0^{\pi/4} \sin^4 4x \; \cos^3 2x \; dx$

7. $\displaystyle\int_0^{\pi} x \sin^7 x \; \cos^2 x \; dx$

8 $\displaystyle\int_0^{\blacksquare} \frac{\sqrt{1-\cos x}}{1+\cos x} \sin^2 x \; dx$

9. $\displaystyle\int_0^{1} x^5 \sin^{-1} x \; dx$

10. $\displaystyle\int_0^{a} x^2 (a^2 - x^2)^{3/2} \; dx$

11. $\displaystyle\int_0^{a} \frac{x^4}{\sqrt{a^2 - x^2}} \; dx$

12. $\displaystyle\int_0^{1} x^4 (1 - x^2)^{3/2} \; dx$

13. $\displaystyle\int_0^{1} \frac{x^3}{(1+x^2)^4} \; dx$

14. $\displaystyle\int_0^{\infty} \frac{x}{(a^2+x^2)^5} \; dx$

15. $\displaystyle\int_0^{\infty} \frac{x^6}{(1+x^2)^{9/2}} \; dx$

ANSWERS

1. $8\pi/15$	2. $8/45$	3. $5\pi/64$
4. $3\pi^2/512$	5. $3\pi/128$	6. $128/1155$
7. $16\pi/315$	8. $8\sqrt{2}/3$	9. $11\pi/192$
10. $\pi a^6/32$	11. $3\pi a^4/16$	12. $3\pi/256$
13. $1/12$	14. $1/12a^6$	15. $1/7$

5.4 Tracing of Curves

The equation $y = f(x)$ in the explicit form geometrically represents a curve. To draw this curve, the basic procedure is to take some values for x and find the corresponding values of y. The set of points (x, y) so tabulated are joined by a smooth curve if the points are non collinear. But we cannot employ this fundamental procedure if the equation of the curve is in the implicit form $f(x, y) = 0$ and it is complicated too.

This topic gives an insight to the process of finding the shape of a plane curve based on its equation by examining certain features. Based on these features we can draw a rough sketch of the curve. **It is highly essential to know the shape of the curve to find its area, length, surface area and volume of solids.**

$f(x, y) = 0$

1. *Symmetry* If the given equation has even powers of x only then the curve is symmetrical about the y axis and if the given equation has even powers of y only then the curve is symmetrical about the x-axis.

If $f(x, y) = f(y, x)$ then the curve is symmetrical about the line $y = x$. Also if $f(x, y) = f(-x, -y)$ then the curve is symmetrical about the origin.

2. *Special points on the curve* : If $f(0, 0) = 0$ then the curve passes through the origin. In such a case we can find the equations of the tangents at the origin by equating the groups of lowest degree terms in x and y to zero.

The points of intersection of the curve with the x axis is got by putting $y = 0$ and that with the y-axis is got by putting $x = 0$.

3. *Asymptotes* Asymptote of a given curve is defined to be the tangent to the given curve at infinity. In otherwords these are lines touching the curve at infinity. Equating the coefficient of highest degree terms in x to zero we get asymptotes parallel to the x-axis and equating the coefficient of highest degree terms in y to zero we get asymptotes parallel to the y-axis.

4. *Region of existence* Region of existence can be determined by finding out the set of permissible (*real*) values of x and y. The curve doesnot lie in the region whenever x or y is imaginary.

By examining these features we can draw a rough sketch of the curve.

Note *In the case of a parametric curve, $x = x(t)$ and $y = y(t)$, we need to vary the parameter t suitably to take a note of the variations in x and y so that the curve can be drawn accordingly.*

List of important points to be examined for tracing a polar curve $f(r, \theta) = 0$

1. *Symmetry* If $f(r, \theta) = f(r, -\theta)$ then the curve is symmetrical about the initial line $\theta = 0$ and $\theta = \pi$.

If $f(r, \theta) = f(r, \pi - \theta)$ then the curve is symmetrical about the line $\theta = \pi/2$ *(positive y- axis)*

If $f(r, \theta) = f(r, \pi/2 - \theta)$ then the curve is symmetrical about the line $\theta = \pi/4$ *(the line $y = x$)*

If $f(r, \theta) = f(r, 3\pi/2 - \theta)$ then the curve is symmetrical about the line $\theta = 3\pi/4$ *(the line $y = -x$)*

If $f(r, \theta) = f(-r, \theta)$ then the curve is symmetrical about the pole *(origin)*

2. *Curve passing through the pole* : If $r = 0$ gives a single value of θ say θ_1 between 0 and 2π then the curve passes through the pole once. $\theta = \theta_1$ is a tangent to the curve at the pole. If it gives two values then the curve passes through the pole twice.

3. *Asymptote* : If $r \to \infty$ as $\theta \to \theta_0$ then the line $\theta = \theta_0$ is an asymptote.

4. *Region of existence* : If r is imaginary for $\theta \in (\alpha, \beta)$ i.e. $\alpha < \theta < \beta$ then the curve doesnot exist in the region between $\theta = \alpha$ and $\theta = \beta$.

5. *Special points* : We can tabulate a set of values of r for convenient values of θ. These give some specific points through which the curve passes.

By examining these features we can draw a rough sketch of the curve.

EXAMPLES

⇒ We have $y^2(a - x) = x^3$ *[This curve is known as cissoid.]*

We observe the following features of the curve.

1. *Symmetry* : The equation contain even powers of y

⇒ **the curve is symmetrical about the x- axis.**

2. *Special points* : The curve passes through $(0, 0)$.

The given equation is $ay^2 - xy^2 = x^3$.

The lowest degree term is ay^2 and $ay^2 = 0 \Rightarrow y = 0$ which is the equation of the x-axis. Hence x-axis is the tangent to the curve at the origin.

Putting $y = 0$ we get $x = 0$ and vice- versa. This means that the curve meets the x-axis and y-axis at the origin only.

3. *Asymptotes* : Equating the coefficient of the highest degree term in y i.e. coefficient of y^2 being $a - x$ to zero we get $x = a$ which is a line parallel to the y-axis. Hence $x = a$ is an asymptote. Also coefficient of the highest degree term in x is x^3 whose coefficient is 1. This implies that there is no asymptote parallel to the x-axis.

4. *Region of existence* : $y^2 = x^3/(a - x)$

$y = \sqrt{x^3/(a - x)}$. This is positive if $x > 0$, $a - x > 0$ or $x < 0$, $a - x < 0$ i.e. $x > 0$, $x < a$, $x < 0$, $x > a$. Since $a > 0$ the second case is not possible. Hence y is real

$x > 0$ and $x < a$ which implies that the curve lies in the interval $0 < x < a$. Further x increases y also increases.

e shape of the curve is as follows.

te *Since the curve meets the coordinate axes at the origin only, the origin is called a sp' with x-axis as the common tangent.*

$$y^2 (a - x) = x^2 (a + x) \quad [\textit{This curve is known as Strophoid}]$$

observe the following features of the curve.

Symmetry The equation contain even powers of y

the curve is symmetrical about the x-axis.

Special points The curve passes through the origin. The equation of the curve can put in the form

$$a (y^2 - x^2) - x y^2 - x^3 = 0.$$

ating the lowest degree terms to zero we have $a (y^2 - x^2) = 0$

nce $y = \pm x$ are the tangents to the curve at the origin. Since there are two tangents origin is called a '*node*'.

xt, putting $y = 0$ we get $x^2 (a + x) = 0 \Rightarrow x = 0, x = -a$.

points are $(0, 0)$ and $(-a, 0)$

o putting $x = 0$ we get $ay^2 = 0$ or $y = 0$ and the point is $(0, 0)$

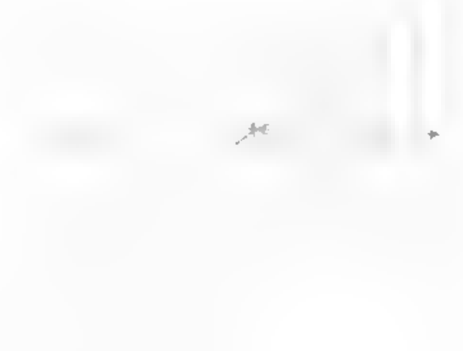
nce we say that the curve intersects the x axis at $(0, 0)$ and $(-a, 0)$ and rsects the y- axis at $(0, 0)$ only

Asymptotes The coefficient of the highest degree in x being x^3 is -1 and ce there is no asymptote parallel to the x axis. Also the coefficient of the highest ree in y being $a - x$, $a - x = 0$ gives $x = a$. Hence $x = a$ is the only asymptote ch is a line parallel to the y-axis.

4. *Region of existence* : $y = \sqrt{x^2 (a+x)/a-x}$.

When $a+x<0$ and $a-x>0$; $a+x>0$ and $a-x<0$, y is imaginary. Also when $a+x<0$ and $a-x<0$ y is not imaginary.

Hence we can say that the curve lies between the lines $x = -a$ and $x = +a$
The shape of the curve is as follows.



>> We observe the following features of the curve

1. *Symmetry* : The equation contain even powers of y and hence the curve is symmetrical about the x-axis.

2. *Special points* : The curve doesnot pass through the origin

If $y = 0$ then $x = a$. The curve meets the x-axis at $(a, 0)$ and it doesnot meet the y-axis.

3. *Asymptotes* : The equation of the curve is $xy^2 - a^3 + a^2x = 0$. Coefficient of y^2 is x and $x = 0$ being the y axis is an asymptote.
Also the coefficient of x is $y^2 + a^2$ and $y^2 + a^2 = 0$ implies that y is imaginary. Hence there is no asymptote parallel to the x-axis

4. *Region of existence* : $y^2 = a^2 (a-x)/x$ $\therefore$ $y = a\sqrt{(a-x)/x}$

y is positive if $a-x>0$ and $x>0$ or $0<x<a$

Hence the curve lies between $x = 0$ and $x = a$.

The shape of the curve is as follows.

We observe the following features of the curve

$f(r,\theta) \neq f(r,-\theta) \Rightarrow$ the curve is not symmetrical about the initial line

$f(r,\theta) \neq f(-r,\theta) \Rightarrow$ the curve is not symmetrical about the pole

$f(r,\theta) = f(r,\pi-\theta) \Rightarrow$ the curve is symmetrical about the line $\theta = \pi/2$

$r = 0$ gives $\sin 3\theta = 0 \Rightarrow 3\theta = n\pi$ or $\theta = n\pi/3$

ing values for $n = 0, 1, 2, \ldots 6$ we get the corresponding values of θ 0, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, 2π and the curve passes through the pole for these values imes.

$0 < \theta < \pi/6$, r is positive and $r = a$ if $\theta = \pi/6$

$\pi/6 < \theta < \pi/3$, r is positive and $r = 0$ if $\theta = \pi/3$

$\pi/3 < \theta < \pi/2$, r is negative and $r = -a$ if $\theta = \pi/2$

e observations implies that r increases from 0 to a as θ varies from 0 to r decreases from a to 0 as θ varies from $\pi/6$ to $\pi/3$,

creases numerically from 0 to a as θ varies from $\pi/3$ to $\pi/2$

ner $f(r,\pi/3-\theta) = f(r,\theta)$ implies that the curve is symmetrical about the line $\pi/6$ so that we conclulde that there is a loop between the lines $\theta = 0$ and $\pi/3$.

arly we can examine the path of the curve as θ moves from $\pi/2$ to π and also π to 2π.

s tabulate a set of values of r corresponding to some values of θ

0	30	60	90	120	150	180	210	240	270	300	330	360
0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

The curve is symmetrical about $\theta = 5\pi/6$ and $3\pi/2$

The shape of the curve is as follows.

5. [illegible]

>> We observe the following features of the curve.

$f(r, \theta) = f(r, -\theta) \Rightarrow$ the curve is symmetrical about the initial line

$f(r, \theta) = f(-r, \theta) \Rightarrow$ the curve is symmetrical about the pole

$r = 0$ gives $a^2 \cos 2\theta = 0$

ie., $\cos 2\theta = 0 \Rightarrow 2\theta = \pi/2$ and $3\pi/2$

$\therefore$ $\theta = \pi/4$ and $\theta = 3\pi/4$ are the tangents to the curve at the pole

When $\theta = 0, r^2 = a^2$ or $r = \pm a$.

Hence the curve meets the initial line at the points $(+a, 0)$ and $(-a, 0)$

Since the curve is symmetrical about the initial line it is composed of two loops. r is real for $\theta \in [0, \pi/4]$ and $[3\pi/4, \pi]$. Also r doesnot tend to infinity for any θ and hence there are no asymptotes.

The shape of the curve is as follows.

pplications of Integral Calculus

results connected with the derivative of the arc length [*Refer unit - II*] will be useful
he discussion of finding the area, length / perimeter of plane curves, the surface
of the revolution of the curve about a given line. Further we also discuss the
ume of a solid of revolution. The relevent formulae for finding these are as follows

Area The area (A) bounded by a curve $y = f(x)$, the x-axis and the
nates $x = a$ and $x = b$ is given by

$$A = \int_{x=a}^{b} y\, dx$$

area (A) between the curves $y = f(x)$ and $y = g(x)$ between $x = a$ and
b is given by

$$A = \int_{a}^{b} f(x)\, dx - \int_{a}^{b} g(x)\, dx.$$

area (A) called the sectorial area bounded by a polar curve $r = f(\theta)$ and the
$\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$$

Length The length of the arc of a curve between two specified points on it for
ous types of curves are given by the following formulae

a process is called *rectification* and the entire length of the curve is called as the
eter of the curve.

artesian curve $y = f(x)$ *or* $x = f(y)$

$$s = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad \int_{y=c}^{d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

arametric curve $x = x(t),\ y = y(t)$

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(iii) *Polar curve* $r = f(\theta)$

$$s = \int_{\theta=\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{or} \quad s = \int_{r=r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr$$

3. ***Surface area*** When a curve revolves about the x - axis a surface is generated and the same is called a surface of revolution. If a curve is bounded by the ordinates $x = a$ and $x = b$ revolves once completely about the x-axis, the area of the surface (S) generated is given by

$$S = \int_{x=a}^{b} 2\pi y\, ds = \int_{a}^{b} 2\pi y \frac{ds}{dx}\, dx$$

where $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

Similarly the surface area of revolution about the y - axis is given by

$$S = \int_{y=c}^{d} 2\pi x\, ds = \int_{c}^{d} 2\pi x \frac{ds}{dy}\, dy\,,$$

where $\dfrac{ds}{dy} = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}$

In the case of a polar curve the surface area of revolution about the initial line is given by

$$S = \int_{\theta=\theta_1}^{\theta_2} 2\pi r \sin\theta\, ds = 2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \frac{ds}{d\theta}\, d\theta$$

where $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

4. ***Volume of revolution*** The volume (V) of the solid generated by the revolution of the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$, about the x - axis is given by

$$V = \pi \int_{x=a}^{b} y^2\, dx$$

Similarly if the axis of revolution is the y - axis, the volume of the solid is given by

$$V = \pi \int_{y=c}^{d} x^2\, dy$$

Also in the case of a polar curve $r = f(\theta)$ the volume (V) of the solid generated is given by

$V = \frac{2\pi}{3} \int r^3 \sin\theta \; d\theta$ *(revolution about the initial line)*

$V = \frac{2\pi}{3} \int r^3 \cos\theta \; d\theta$ *(revolution about the line $\theta = \pi/2$)*

Applications formulae at a glance

	Cartesian curve	Parametric curve	Polar curve
Area (A)	$\int_a^b y \, dx$ or $\int_c^d x \, dy$	$\int_{t_1}^{t_2} y \frac{dx}{dt} \, dt$ or $\int_{t_1}^{t_2} x \frac{dy}{dt} \, dt$	$\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta$
Length (s)	$\int_a^b \frac{ds}{dx} \, dx$ or $\int_c^d \frac{ds}{dy} \, dy$	$\int_{t_1}^{t_2} \frac{ds}{dt} \, dt$	$\int_{\theta_1}^{\theta_2} \frac{ds}{d\theta} \, d\theta$ or $\int_{r_1}^{r_2} \frac{ds}{dr} \, dr$
Surface area of revolution (S)	$2\pi \int_a^b y \frac{ds}{dx} \, dx$ (about the x-axis) $2\pi \int_c^d x \frac{ds}{dy} \, dy$ (about the y-axis)	$2\pi \int_{t_1}^{t_2} y \frac{ds}{dt} \, dt$ (about the x-axis) $2\pi \int_{t_1}^{t_2} x \frac{ds}{dt} \, dt$ (about the y-axis)	$2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \frac{ds}{d\theta} \, d\theta$
Volume of revolution (V)	$\pi \int_a^b y^2 \, dx$ (about the x-axis) $\pi \int_c^d x^2 \, dy$ (about the y-axis)	$\pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt} \, dt$ (about the x-axis) $\pi \int_{t_1}^{t_2} x^2 \frac{dy}{dt} \, dt$ (about the y-axis)	$\frac{2\pi}{3} \int r^3 \sin\theta \; d\theta$ (about the line $\theta = 0$ or x-axis) $\frac{2\pi}{3} \int r^3 \cos\theta \; d\theta$ (about the line $\theta = \pi/2$ or y-axis)

5.51 Application related to standard curves

1. ***The Astroid*** . Astroid is the curve represented by the equation .

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Its parametric equation is $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

We shall find its shape first and then determine the associated area, perimeter, su area and the volume.

We tabulate x, y corresponding to certain angles of θ in the interval $[0, 2\pi$

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	a	0	$-a$	0	a
y	0	a	0	$-a$	0

From the table we conclude that the curve meets the x-axis at the points $(a, 0)$ $(-a, 0)$. Also it meets the y axis at the points $(0, a)$ and $(0, -a)$. S $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$, we have $|x| \le a$ and $|y| \le a$. Hence we infer the entire curve lies within a circle of radius 'a' having origin as the centre.

Also we have from the cartesian equation of the curve,

$$f(x, y) = f(-x, y);\ f(x, y) = f(x, -y);\ f(x, y) = f(y, x)$$

Hence the curve is symmetrical about the coordinate axes and also about the line $y = x$.

Taking a note of the values of x and y as θ advances from one quadrant to the other the shape of the curve is as shown.

WORKED PROBLEMS

31. *Find the area enclosed by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$*

Note : *In any problem on applications we need to draw the curve first by briefly examining important features.*

The curve astroid is symmetrical about the coordinate axes and hence the requ area(A) is equal to four times the area in the first quadrant

i.e., $$A = 4\int_0^a y\,dx = 4\int_0^a y\,\frac{dx}{d\theta}\,d\theta$$

We have $x = a\cos^3\theta$, $y = a\sin^3\theta$ $\therefore$ $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

When $x = 0 : a\cos^3\theta = 0$ or $\cos^3\theta = 0 \Rightarrow \theta = \pi/2$

$x = 1 : a\cos^3\theta = a$ or $\cos^3\theta = 1 \Rightarrow \theta = 0$

$$\therefore \quad A = 4\int_{\theta=\pi/2}^{0} a\sin^3\theta\,(-3a\cos^2\theta\,\sin\theta)\,d\theta$$

$$= 12a^2\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta$$

$$= 12a^2\,\frac{(3)(1)\,(1)}{6\times4\times2}\cdot\frac{\pi}{2}, \text{ by reduction formula}$$

Thus the area enclosed (A) is $3\pi a^2/8$ sq. units

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32. *Find the perimeter of the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$

>> Since the curve is symmetrical about the coordinate axes, the perimeter (*entire length*) of the curve is four times its length in the first quadrant

$$\therefore \quad l = 4\int_{\theta=0}^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{9a^2\cos^4\theta\,\sin^2\theta + 9a^2\sin^4\theta\,\cos^2\theta}\,d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{9a^2\cos^2\theta\,\sin^2\theta(\cos^2\theta+\sin^2\theta)}\,d\theta$$

$$= 4\int_0^{\pi/2} 3a\cos\theta\,\sin\theta\,d\theta$$

$$= 6a\int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$= 6a\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = -3a(\cos\pi - \cos 0) = -3a(-1-1) = 6a$$

Thus the perimeter of the curve is $6a$ units.

33. *Find the surface area [illegible] of the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ *about the* [illegible]

→ Because of symmetry the required surface area is equal to twice the surface a[illegible] by the revolution of the first quadrant of the curve

$$S = 2 \times \int_0^a 2\pi y \, ds = 4\pi \int_{\theta = 0}^{\pi/2} y \frac{ds}{d\theta} \, d\theta$$

But $\quad \dfrac{ds}{d\theta} = \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2} = 3a \cos\theta \sin\theta \cdots$ *(Refer Ex - 32)*

$$\text{Hence } S = 4\pi \int_0^{\pi/2} a \sin^3\theta \cdot 3a \cos\theta \sin\theta \, d\theta$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4\theta \cos\theta \, d\theta$$

$$= 12\pi a^2 \cdot \frac{(3)}{5 \times 3 \times 1} \text{ by reduction formula.}$$

Thus the required surface area = $12\pi a^2/5$ sq.units.

34. *Find the volume of the solid generated by the revolution of the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ *about the x-axis*

→ Because of symmetry the required volume (V) is equal to twice the volume of [illegible] solid generated by the curve in the first quadrant about the x-axis

$$\therefore \quad V = 2 \times \int_0^a \pi y^2 \, dx = 2\pi \int_0^a y^2 \frac{dx}{d\theta} \, d\theta$$

$$= 2\pi \int_{\theta = \pi/2}^{0} a^2 \sin^6\theta \, (-3a \cos^2\theta \sin\theta) \, d\theta$$

$$= 6\pi a^3 \int_0^{\pi/2} \sin^7\theta \cos^2\theta \, d\theta$$

$$= 6\pi a^3 \frac{(6)(4)(2)\cdot(1)}{9 \times 7 \times 5 \times 3 \times 1} \text{ by reduction formula.}$$

Thus the required volume of the solid is $32\pi a^3/105$ cubic units.

2 *Cycloid* Cycloid is a curve generated by a point on the circumference of a circle which rolls on a fixed straight line known as the base. Imagine a wheel rolling on a straight line without slipping. A fixed point on the rim of the wheel traces the cycloid. The parametric equation of the cycloid can be in the following four forms.

(i) $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$

(ii) $x = a(\theta + \sin\theta),\ y = a(1-\cos\theta)$

(iii) $x = a(\theta - \sin\theta),\ y = a(1+\cos\theta)$

(iv) $x = a(\theta + \sin\theta),\ y = a(1+\cos\theta)$

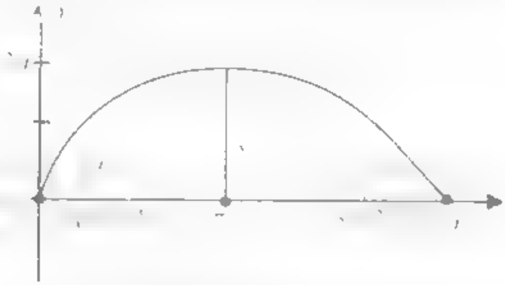
Tracing of the cycloid : $\boldsymbol{x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)}$.

>> Let us tabulate x, y for certain values of θ in the interval $[0, 2\pi]$ where θ is in radians.

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a(\pi/2-1)$	$a\pi$	$a(3\pi/2+1)$	$2a\pi$
y	0	a	$2a$	a	0

From the table we can conclude that the curve intersects the x-axis at $x = 0$ and $2a\pi$. Also, we have $y = a(1-\cos\theta)$ and since $|\cos\theta| \le 1$, y is non negative. Hence the curve lies above the x-axis.

Taking a note of the values of x and y as θ advances in the interval $[0, 2\pi]$ the shape of the curve is as follows. It is called an arch of the curve.



WORKED PROBLEMS

35. *Find the area of an arch of the cycloid* $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$

>> Area $$A = \int_{\theta=0}^{2\pi} y\,\frac{dx}{d\theta}\,d\theta$$

ie., $$A = \int_{0}^{2\pi} a(1-\cos\theta)\cdot a(1-\cos\theta)\,d\theta$$

$$= a^2 \int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{2\pi} 4\sin^4(\theta/2) \, d\theta$$

Put $\theta/2 = t$ $\quad \therefore \quad d\theta = 2dt,\ t$ varies from 0 to π

$$\therefore \qquad A = 8a^2 \int_{t=0}^{\pi} \sin^4 t \, dt = 8a^2 \cdot 2 \int_0^{\pi/2} \sin^4 t \, dt$$

ie., $\quad A = 16a^2 \, \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}$ by reduction formula

Thus the area enclosed by an arch of the curve on its base is $3\pi a^2$ sq.units

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36. *Find the length of an arch of the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

>> Length $(l) = \displaystyle\int_{\theta=0}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$

ie., $\quad l = \displaystyle\int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta} \, d\theta$

$$\int_0^{2\pi} a \sqrt{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} \, d\theta$$

$$= a \int_0^{2\pi} \sqrt{2(1-\cos\theta)} \, d\theta = a \int_0^{2\pi} \sqrt{2 \cdot 2\sin^2(\theta/2)} \, d\theta$$

$$= 2a \int_0^{2\pi} \sin(\theta/2) \, d\theta = -\left[\frac{2a\cos(\theta/2)}{1/2}\right]_0^{2\pi}$$

$$= -4a(\cos\pi - \cos 0) = -4a(-1-1) = 8a$$

Thus the required length is $8a$

37. [illegible]
$x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ *about the x-axis*

>> Surface area $S = 2\pi \int_0^{2\pi} y \dfrac{ds}{d\theta}\, d\theta$

But $\dfrac{ds}{d\theta} = \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2} = 2a \sin(\theta/2)$ *[Refer Ex-36]*

$$S = 2\pi \int_0^{2\pi} a(1-\cos\theta) \cdot 2a \sin(\theta/2)\, d\theta$$

$$= 4\pi a^2 \int_0^{2\pi} 2\sin^3(\theta/2)\, d\theta = 8\pi a^2 \int_0^{2\pi} \sin^3(\theta/2)\, d\theta$$

Put $\theta/2 = t$ $\therefore$ $d\theta = 2dt$ and t varies from 0 to π.

Hence $S = 8\pi a^2 \int_{t=0}^{\pi} \sin^3 t \cdot 2dt = 16\pi a^2 \cdot 2 \int_0^{\pi/2} \sin^3 t\, dt$

i.e., $S = 32\pi a^2 \cdot \dfrac{2}{3}$ by applying reduction formula

Thus the required surface area is $64\pi a^2/3$ sq.units.

38. *Find the volume of the solid generated by the revolution of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about its base*

>> $V = \pi \int_0^{2\pi} y^2 \dfrac{dx}{d\theta}\, d\theta$

$$= \pi \int_0^{2\pi} a^2(1-\cos\theta)^2 \cdot a(1-\cos\theta)\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} \{2\sin^2(\theta/2)\}^3\, d\theta = 8\pi a^3 \int_0^{2\pi} \sin^6(\theta/2)\, d\theta$$

$$= 8\pi a^3 \cdot 2 \int_0^{\pi} \sin^6(\theta/2)\, d\theta = 16\pi a^3 \int_0^{\pi} \sin^6(\theta/2)\, d\theta$$

Put $\theta/2 = t$ $\therefore$ $d\theta = 2dt$ and t varies from 0 to $\pi/2$

$$V = 16\pi a^3 \int_0^{\pi/2} \sin^6 t \cdot 2\, dt$$

$$32\pi a^3 \ \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \text{ by reduction formula.}$$

Thus the required volume is $5\pi^2 a^3$ cubic units.

3 *Cardioide*

Tracing of the cardioide : $r = a(1+\cos\theta)$

We observe the following features of the curve

(i) $f(r, \theta) = f(r, -\theta)$ and hence the curve is symmetrical about the init

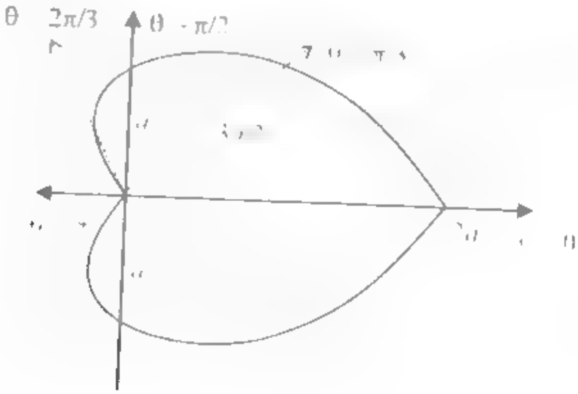
(ii) When $\theta = \pi$ $r = 0$ and hence the curve passes through the pole θ = tangent to the curve at the pole.

(iii) Since $|\cos\theta| \le 1$, $|r| \le 2a$ and hence the curve lies within the circle o $2a$ having its centre at the pole.

Let us tabulate r for certain angles of θ

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$
r	$2a$	$3a/2$	a	$a/2$

It is evident that as θ increases from 0 to π, r decreases from $2a$ to 0 Th shape of the curve is as follows.



WORKED PROBLEMS

39. *Find the area of the cardioide* $r = a(1 + \cos\theta)$

>> Since the curve is symmetrical about the initial line, the total area (A) is twice the area above the initial line.

ie., $$A = 2 \cdot \frac{1}{2} \int_0^{\pi} r^2 \, d\theta = \int_0^{\pi} a^2 (1 + \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} [2\cos^2(\theta/2)]^2 \, d\theta = 4a^2 \int_0^{\pi} \cos^4(\theta/2) \, d\theta$$

Put $\theta/2 = t$ $\therefore$ $d\theta = 2dt$ and t varies from 0 to $\pi/2$

$\therefore$ $$A = 4a^2 \int_{t=0}^{\pi/2} \cos^4 t \cdot 2dt = 8a^2 \int_0^{\pi/2} \cos^4 t \, dt$$

$$= 8a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$, by reduction formula.

Thus the area enclosed is $3\pi a^2/2$ sq.units.

---

40. *Find the perimeter of the cardioide* $r = a(1 + \cos\theta)$

>> Perimeter (*length*) = 2 (length of the upper half of the curve)

ie., $$= 2 \int_0^{\pi} \frac{ds}{d\theta} \, d\theta \quad \text{where} \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Now $$\frac{ds}{d\theta} = \sqrt{a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta} = a\sqrt{2(1 + \cos\theta)}$$

$$= 2a \cos(\theta/2)$$

$\therefore$ $$\text{perimeter} = 2 \int_0^{\pi} 2a \cos(\theta/2) \, d\theta = 4a \left[\frac{\sin(\theta/2)}{1/2}\right]_0^{\pi} = 8a$$

Thus the perimeter of the curve is $8a$ units.

11. *Find the surface area of the revolution of the cardioid about the initial line*

>> Surface area $S = \int 2\pi r \sin\theta \dfrac{ds}{d\theta}\, d\theta$

But $\quad \dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} = 2a \cos(\theta/2) \qquad$ *(Refer Example 40)*

$$\therefore \qquad S = 2\pi \int_{\theta=0}^{\pi} a(1+\cos\theta)\sin\theta \; 2a\cos(\theta/2)\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 2\cos^2(\theta/2)\cdot 2\sin(\theta/2)\cos(\theta/2)\cos(\theta/2)\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4(\theta/2)\sin(\theta/2)\, d\theta$$

Put $\quad \theta/2 = t \quad \therefore \quad d\theta = 2dt$ and t varies from 0 to $\pi/2$

Hence $\quad S = 16\pi a^2 \int_{t=0}^{\pi/2} \cos^4 t \, \sin t \cdot 2\, dt$

$$32\pi a^2 \cdot \frac{(3)(1)}{5\times 3} \quad \text{by reduction formula.}$$

Thus the required surface area is $32\pi a^2/5$ sq.units.

12. [illegible] $r = a(1+\cos\theta)$ [illegible]

$$>> \qquad V = \frac{2}{3}\pi \int_0^{\pi} r^3 \sin\theta\, d\theta = \frac{2\pi}{3}\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta\, d\theta$$

Put $\quad t = 1 + \cos\theta \quad \therefore \quad dt = -\sin\theta\, d\theta$

If $\quad \theta = 0,\ t = 2$ and if $\theta = \pi,\ t = 0$

$$\therefore \qquad V = \frac{2\pi a^3}{3}\int_2^0 t^3(-dt) = \frac{2\pi a^3}{3}\int_0^2 t^3\, dt$$

$$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 = \frac{2\pi a^3}{3}(4-0) = \frac{8\pi a^3}{3}$$

Thus the required volume is $8\pi a^3/3$ cubic units.

Unit - VI

DIFFERENTIAL EQUATIONS

6.1 Introduction

Many problems in all branches of science and engineering when analysed for putting in a mathematical form assumes the form of a *differential equation* An engineer or an applied mathematician will be mostly intrested in obtaining a solution for the associated equation without bothering much on the rigorous aspects like the proof, validity conditions, region of existence etc Accordingly the study of differential equations at various levels is focussed on the methods of solving the equations

6.2 Preliminaries

Ordinary Differential Equation (O.D.E)

If $y = f(x)$ is an unknown function, an equation which involves atleast one derivative of y *w.r.t* x is called an *ordinary differential equation* which in future will be simply referred to as a *Differential Equation (D E)*

The *order* of the D E is the order of the highest derivative present in the equation and the *degree* of the D E is the degree of the highest order derivative after clearing the fractional powers.

Finding y as a function of x explicitly $[y = f(x)]$ or a relationship in x and y satisfying the D E $[f(x, y) = c]$ constitutes the solution of the D E

Observe the following *examples* along with their order and degree

1. $\dfrac{dy}{dx} = 2x$ [order = 1, degree = 1]

2. $\left(\dfrac{dy}{dx}\right)^2 + 3\left(\dfrac{dy}{dx}\right) + 2 = 0$ [order = 1, degree = 2]

3. $\dfrac{d^2x}{dt^2} + w^2 x = 0$ [order = 2, degree = 1]

4. $\dfrac{d^3y}{dx^3} + 5\left(\dfrac{d^2y}{dx^2}\right) + \left(\dfrac{dy}{dx}\right)^3 = \sin x$ [order = 3, degree = 1]

5. $\left(r^2 + r_1^2\right)^{3/2} = r^2 + 2\,r_1^2 - r\,r_2$ where $r_1 = \dfrac{dr}{d\theta}, \quad r_2 = \dfrac{d^2 r}{d\theta^2}$

Here we need to clear the fractional power 3 2 by squaring both sides

$$\therefore \qquad (r^2 + r_1^2)^3 = \left(r^2 + 2\,r_1^2 - r\,r_2\right)^2$$

Observing the term r_2^2 in the R H S we conclude that the order of the equation and the degree is 2.

General solution and particular solution

A solution of a D E is a relation between the dependent and independent varia satisfying the given equation identically.

For example $y = e^{3x}$ is a solution of the D.E

$$\frac{dy}{dx} - 3\,e^{3x} = 0 \text{ as we could see that}$$

$$\frac{d}{dx}\,(e^{3x}) - 3\,e^{3x} = 3\,e^{3x} - 3\,e^{3x} = 0$$

Also $\quad x = 2\cos wt + 3\sin wt$ is a solution of the D.E

$$\frac{d^2 x}{dt^2} + w^2 x = 0 \text{ as we have,}$$

$$\frac{d^2}{dt^2}\left(2\cos wt + 3\sin wt\right) + w^2\left(2\cos wt + 3\sin wt\right)$$

$$= -2w^2\cos wt - 3w^2\sin wt + 2w^2\cos wt + 3w^2\sin wt = 0$$

Now let us look at the first of the above example in the reverse way
That is to consider the D.E

$$\frac{dy}{dx} - 3\,e^{3x} = 0 \text{ and try to find } y.$$

i e $\quad \dfrac{dy}{dx} = 3\,e^{3x}$ and to find y we need to get rid off $\dfrac{d}{dx}$ being the different operator Obviously we have to use anti differentiation, that is integration

$$\frac{dy}{dx} = 3\,e^{3x}$$

$$\Rightarrow \qquad y = \int 3\,e^{3x}\,dx + c \quad ie., \quad y = 3 \cdot \frac{e^{3x}}{3} + c$$

or $\qquad y = e^{3x} + c,\ c$ being the constant of integration.

Integration is directly or indirectly involved in the process of getting a solution of the given D.E and accordingly the solution will be involved with arbitrary constants. Such a solution is called as the **general solution** of the D.E. It is obvious that the number of arbitrary constants present in the solution is equal to the order of the D.E.

n the first example we verified that e^{3x} is a solution of the equation $y' - 3e^{3x} = 0$. ve obtained $y = e^{3x} + c$ as the solution by solving the equation, c being arbitrary. he solution $y = e^{3x}$ is a particular case of the solution $e^{3x} + c$ (for $c = 0$) and ccordingly it is called a particular solution and the same can be interpreted as follows.

the arbitrary constants present in the solution are evaluated by using a set of given nditions then the solution so obtained is called a **particular solution**. In many ysical problems these conditions can be formulated from the problem itself.

the discussion of the first example, let us consider the initial condition $y(0) = 1$ at is $y = 1$ when $x = 0$.

e general solution $y = e^{3x} + c$ becomes

$$1 = e^{0} + c \quad \text{or} \quad 1 = 1 + c \quad \text{which gives} \quad c = 0.$$

us $y = e^{3x}$ is a particular solution of the D.E. In this context we can also say that solution of an O.D.E is unique in the sense that, the solution can differ only by stants. It may be just noted that the general solution of the D.E

$$\frac{d^2 x}{dt^2} + w^2 x = 0 \quad \text{is} \quad x = c_1 \cos wt + c_2 \sin wt$$

pose we impose the conditions $x(0) = 2$ and $x'(0) = 3w$ which means that 2, $\frac{dx}{dt} = 3w$ when $t = 0$, then we obtain $c_1 = 2$ and $c_2 = 3$

equivalent to saying that $x = 2\cos wt + 3\sin wt$ is a particular solution of the $\frac{d^2 x}{dt^2} + w^2 x = 0$.

s chapter we first discuss various methods of solving differential equations of first and first degree.

: *Basic integration and integration methods are **essential prerequisites** for this chapter*

6.3 Solution of differential equations of first o and first degree

Recollecting the definition of the order and the degree of a D E, a first or degree equation will be of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)\,dx + N(x, y)\,dy = 0.$$

The mainly classified four types of differential equations of first order and are as follows :

1. **Variables separable equations**
2. **Homogeneous equations**
3. **Exact equations**
4. **Linear equations**

[illegible]

If the given D E can be put in the form such that the coefficient of dx is a f the variable x only and the coefficient of dy is a function of y only then equation is said to be in the separable form
The modified form of such an equation will be,

$$P(x)\,dx + Q(y)\,dy = 0. \quad \textbf{Integrating we have}$$

$$\int P(x)\,dx + \int Q(y)\,dy = c$$

This is the general solution of the equation

ILLUSTRATIVE EXAMPLES

[illegible]

>> $\sec^2 x \tan y \; dx + \sec^2 y \tan x \; dy = 0$

Dividing throughout by $\tan y \tan x$ we get

$$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0 \quad \textit{(Variables are separated)}$$

$$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = c$$

ie., $\log(\tan x) + \log(\tan y) = c$

ie., $\log(\tan x \cdot \tan y) = \log k$ *(say)*

Thus $\tan x \tan y = k$ is the required solution

6.3 Solution of differential equations of first o... and first degree

Recollecting the definition of the order and the degree of a D.E, a first or... degree equation will be of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)\,dx + N(x, y)\,dy = 0$$

The mainly classified four types of differential equations of first order and ... are as follows :

1. Variables separable equations
2. Homogeneous equations
3. Exact equations
4. Linear equations

Recapitulation of the method of separation of variables

If the given D.E can be put in the form such that the coefficient of dx is a fu... the variable x only and the coefficient of dy is a function of y only then ... equation is said to be in the separable form
The modified form of such an equation will be,

$$P(x)\,dx + Q(y)\,dy = 0. \quad \text{Integrating we have}$$

$$\int P(x)\,dx + \int Q(y)\,dy = c$$

This is the general solution of the equation.

ILLUSTRATIVE EXAMPLES

>> $\sec^2 x \tan y \; dx + \sec^2 y \tan x \; dy = 0$

Dividing throughout by $\tan y \tan x$ we get

$$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0 \quad \text{(Variables are separated)}$$

$$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = c$$

ie $\log(\tan x) + \log(\tan y) = c$

ie., $\log(\tan x \cdot \tan y) = \log k$ (say)

Thus $\tan x \tan y = k$ is the required solution

Solve $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

$>>\quad \dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

ie., $\quad \dfrac{dy}{dx} = e^{-2y}\,(e^{3x} + x^2)$

or $\quad \dfrac{dy}{e^{-2y}} = (e^{3x} + x^2)\,dx$ by separating the variables

$\Rightarrow\quad \int e^{2y}\,dy - \int (e^{3x} + x^2)\,dx = c$

Thus $\quad \dfrac{e^{2y}}{2} - \dfrac{e^{3x}}{3} - \dfrac{x^3}{3} = c,$ is the required solution.

Solve [illegible]

$>>\quad xy\dfrac{dy}{dx} = 1 + x + y + xy$

ie., $\quad xy\dfrac{dy}{dx} = (1 + x) + y\,(1 + x)$

ie., $\quad xy\dfrac{dy}{dx} = (1 + x)(1 + y)$

or $\quad \dfrac{y\,dy}{1 + y} = \dfrac{1 + x}{x}\,dx$ by separating the variables.

$\Rightarrow\quad \int \dfrac{y}{1 + y}\,dy - \int \dfrac{1 + x}{x}\,dx = c$

or $\quad \int \dfrac{(1 + y) - 1}{1 + y}\,dy - \int \dfrac{1}{x}\,dx - \int 1\,dx = c$

ie., $\quad \int 1\,dy - \int \dfrac{1}{1 + y}\,dy - \log x - x = c$

ie., $\quad y - \log(1 + y) - \log x - x = c$

Thus $\quad (y - x) - \log[\,x\,(1 + y)\,] = c,$ is the required solution

Note *Some differential equations can be reduced to the variables separable form by ta suitable substitution. We identify a few types of equation along with the ass substitutions.*

(i) $\dfrac{dy}{dx} = f(ax + by + c)$; substitution : $t = ax + by + c$

(ii) $\dfrac{dy}{dx} = \dfrac{(ax + by) + c}{k(ax + by) + c'}$; substitution : $t = ax + by$

Solve $\dfrac{dy}{dx} = (9x + y + 1)^2$

>> We have $\dfrac{dy}{dx} = (9x + y + 1)^2$

Put $t = 9x + y + 1 \quad \therefore \quad \dfrac{dt}{dx} = 9 + \dfrac{dy}{dx}$ or $\dfrac{dt}{dx} - 9 = \dfrac{dy}{dx}$

Now (1) becomes $\dfrac{dt}{dx} - 9 = t^2$ or $\dfrac{dt}{dx} = t^2 + 9$

Hence $\displaystyle\int \frac{dt}{t^2 + 3^2} = \int dx + c$ *ie.,* $\dfrac{1}{3}\tan^{-1}\left(\dfrac{t}{3}\right) - x = c$

Thus $\dfrac{1}{3}\tan^{-1}\left(\dfrac{9x + y + 1}{3}\right) - x = c,$ is the required solution

$\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}$

>> We have $\dfrac{dy}{dx} = \dfrac{(x + y) + 1}{2(x + y) + 3}$

Put $t = x + y \quad \therefore \quad \dfrac{dt}{dx} = 1 + \dfrac{dy}{dx}$ or $\dfrac{dt}{dx} - 1 = \dfrac{dy}{dx}$

The given equation becomes,

$$\frac{dt}{dx} - 1 = \frac{t + 1}{2t + 3} \quad \text{or} \quad \frac{dt}{dx} = 1 + \frac{t + 1}{2t + 3}$$

ie., $\dfrac{dt}{dx} = \dfrac{2t + 3 + t + 1}{2t + 3}$ or $\dfrac{dt}{dx} = \dfrac{3t + 4}{2t + 3}$

ie., $\dfrac{2t + 3}{3t + 4}\,dt = dx$

$\Rightarrow \displaystyle\int \frac{2t + 3}{3t + 4}\,dt - \int dx = c$

Let $2t + 3 = l(3t + 4) + m$

or $\quad 2t+3 = (3l)t + (4l+m)$

$\Rightarrow \quad 3l = 2$ and $4l + m = 3$

$l = 2/3$ and $8/3 + m = 3$ or $m = 1/3$.

Thus $\quad 2t+3 = 2/3 \cdot (3t+4) + 1/3$

Hence (1) becomes $\displaystyle\int \frac{2/3 \cdot (3t+4)+1/3}{3t+4}\, dt - \int dx = c$

ie., $\quad \displaystyle\frac{2}{3}\int 1\, dt + \frac{1}{3}\int \frac{dt}{3t+4} - x = c$

ie., $\quad \displaystyle\frac{2t}{3} + \frac{1}{9}\log(3t+4) - x = c$, where $t = x+y$

$$\frac{2(x+y)}{3} + \frac{1}{9}\log(3x+3y+4) - x = c$$

Thus $\quad \displaystyle\frac{1}{3}(2y - x) + \frac{1}{9}\log(3x+3y+4) = c$, is the required solution

6.32 Homogeneous Differential Equations

Definition of homogeneous function

A function $u = f(x, y)$ is said to be a *homogeneous function of degree 'n'* if

$$u = x^n\, g(y/x) \text{ or } u = y^n\, g(x/y)$$

Examples

1. $u = 2x + 3y$

$u = x[2 + 3(y/x)] = x^1 g(y/x)$

$\Rightarrow \quad u$ is a homogeneous function of degree 1.

2. $u = x^2 - xy + y^2$

$u = x^2 [1 - (y/x) + (y/x)^2] = x^2 g(y/x)$

$\Rightarrow \quad u$ is a homogeneous function of degree 2

3. $u = xy^2 + x^2 y$

$u = x^3 [(y/x)^2 + (y/x)] = x^3 g(y/x)$

$\Rightarrow \quad u$ is a homogeneous function of degree 3

Note *In these examples we observe that the total degree of the terms involved in each of the functions are respectively 1, 2, 3 and hence the homogeneous aspect can be recognized instantly*

4. $u = x\cos(y/x) + y\sin(y/x)$

$u = x[\cos(y/x) + (y/x)\sin(y/x)] = x^1 g(y/x)$

$\Rightarrow$ u is homogeneous of degree 1.

5. $u = x^2\tan(x/y) + y^2$

$u = y^2[(x/y)^2\tan(x/y) + 1] = y^2 g(x/y)$

$\Rightarrow$ u is homogeneous of degree 2.

6. $u = x\log y - x\log x$

$u = x(\log y - \log x) = x\log(y/x) = x^1 g(y/x)$

$\Rightarrow$ u is homogeneous of degree 1.

7. $u = y + \sqrt{xy}$

$u = x\left[(y/x) + \frac{1}{x}\sqrt{xy}\right] = x\left[(y/x) + \sqrt{xy/x^2}\right] = x\left[(y/x) + \sqrt{y/x}\right] = x^1 g(y$

$\Rightarrow$ u is homogeneous of degree 1.

8. $u = \sqrt{x^2 + y^2}$

$u = \sqrt{x^2[1 + (y/x)^2]} = x\sqrt{1 + (y/x)^2} = x^1 g(y/x)$

$\Rightarrow$ u is homogeneous of degree 1.

Remark . *These examples gives an insight to recognize the homogeneous aspect of fun along with the associated degree instantly.*

Homogeneous differential equation

A D E of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be a homogeneo if both $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degre

[illegible]

We prefer to have the differential equation in the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$ after recognizing that the D E is a homogeneous one

We take the substitution $y = vx$ so that,

$$\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx} \text{, by product rule.}$$

[*Since both $f(x, y)$ and $g(x, y)$ are expressible in the form $x^n F(y/x)$ it wi feasible to take a substitution for y/x Thus if $v = y/x$ then it is obvious that v is a fun of x as y is a function of x For convenience we take $y = vx$ and then differentiate w*

Substituting for y and $\dfrac{dy}{dx}$ the given equation after simplification converts into an equation which can be solved by separating the variables v and x.

Finally we substitute back for v which is equal to y/x.

If the coefficient of dx and dy involves terms with (x/y) then we have to write the equation associated with $\dfrac{dx}{dy}$ and use the substitution $x = vy$.

This gives $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$. We then proceed as described earlier.

WORKED PROBLEMS

1. *Solve* : $(x^2 - y^2)\,dx = x\,y\,dy$

(*Observe that the coefficient of dx and dy are homogeneous functions of degree* 2)

>> We have $$\frac{dy}{dx} = \frac{x^2 - y^2}{x\,y} \qquad \ldots (1)$$

Put $y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes, $$v + x\frac{dv}{dx} = \frac{x^2 - v^2 x^2}{x \cdot vx}$$

ie., $$v + x\frac{dv}{dx} = \frac{x^2(1 - v^2)}{x^2 v}$$

ie., $$x\frac{dv}{dx} = \frac{1 - v^2}{v} - v \quad \text{or} \quad x\frac{dv}{dx} = \frac{1 - 2v^2}{v}$$

$$\frac{v}{1 - 2v^2}\,dv = \frac{dx}{x}$$, by separating the variables.

$\Rightarrow$ $$\int \frac{v}{1 - 2v^2}\,dv - \int \frac{dx}{x} = c$$

ie., $$-\frac{1}{4}\log(1 - 2v^2) - \log x = c$$

ie., $$\log(1 - 2v^2) + 4\log x = -4c$$

ie., $$\log\left[(1 - 2v^2)\,x^4\right] = \log k \ \ (say) \qquad \because\ 4\log x = \log x^4$$

or $$(1 - 2v^2)\,x^4 = k \ \text{ where } v = y/x$$

ie., $$\left(1 - \frac{2y^2}{x^2}\right)x^4 = k \quad \text{or} \quad x^2(x^2 - 2y^2) = k$$

Thus $x^4 - 2x^2y^2 = k$, is the required solution.

2. *Solve* $x^2 y\,dx - (x^3 + y^3)\,dy = 0$

(Observe that the coefficient of dx and dy are homogeneous functions of degree 3)

>> The given equation can be written as,

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

Put $\quad y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes, $\quad v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3 + v^3 x^3}$

ie., $\quad v + x\frac{dv}{dx} = \frac{x^3 v}{x^3(1 + v^3)} \quad$ or $\quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v$

ie., $\quad x\frac{dv}{dx} = \frac{v - v - v^4}{1 + v^3} \quad$ or $\quad x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$

$\quad \frac{1+v^3}{v^4}\,dv = \frac{-dx}{x}$, by separating the variables.

$\Rightarrow \quad \int \frac{1}{v^4}\,dv + \int \frac{1}{v}\,dv + \int \frac{dx}{x} = c$

ie., $\quad \frac{v^{-3}}{-3} + \log v + \log x = c$

ie., $\quad -\frac{1}{3v^3} + \log(vx) = c$, where $v = \frac{y}{x}$

Thus $\quad \frac{-x^3}{3y^3} + \log y = c$, is the required solution

3. *Solve* $(y^3 - 3x^2 y)\,dx - (x^3 - 3xy^2)\,dy = 0$

>> The given equation can be written as,

$$\frac{dy}{dx} = \frac{y^3 - 3x^2 y}{x^3 - 3xy^2}$$

Put $\quad y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes, $\quad v + x\frac{dv}{dx} = \frac{v^3 x^3 - 3x^2 \cdot vx}{x^3 - 3x \cdot v^2 x^2}$

i.e., $v + x\,\dfrac{dv}{dx} = \dfrac{x^3\,(v^3 - 3v)}{x^3\,(1 - 3v^2)}$ or $x\,\dfrac{dv}{dx} = \dfrac{v^3 - 3v}{1 - 3v^2} - v$

i.e., $x\,\dfrac{dv}{dx} = \dfrac{v^3 - 3v - v + 3v^3}{1 - 3v^2}$

i.e., $x\,\dfrac{dv}{dx} = \dfrac{4\,(v^3 - v)}{1 - 3v^2}$ or $\dfrac{1 - 3v^2}{v^3 - v}\,dv = 4\,\dfrac{dx}{x}$

Hence $\displaystyle\int \frac{1 - 3v^2}{v^3 - v}\,dv - 4\int \frac{dx}{x} = c$

or $\displaystyle -\int \frac{3v^2 - 1}{v^3 - v}\,dv - 4\log x = c$

i.e., $-\log(v^3 - v) - \log x^4 = c$

i.e., $\log\left[(v^3 - v)\,x^4\right] = -c = \log k$ *(say)*

$\therefore$ $(v^3 - v)\,x^4 = k$, where $v = y/x$

$\therefore$ $\left(\dfrac{y^3}{x^3} - \dfrac{y}{x}\right)x^4 = k$ or $x\,y^3 - x^3\,y = k$

Thus $x\,y^3 - x^3\,y = k$, is the required solution.

>> We have $\dfrac{dy}{dx} = \dfrac{y}{x - \sqrt{xy}}$ (1)

(Observe that both the numerator and denominator are homogeneous functions of degree 1)

Put $y = vx$ $\therefore$ $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$

Now (1) becomes, $v + x\,\dfrac{dv}{dx} = \dfrac{vx}{x - \sqrt{x \cdot vx}}$

i.e., $v + x\,\dfrac{dv}{dx} = \dfrac{vx}{x\,(1 - \sqrt{v})}$ or $x\,\dfrac{dv}{dx} = \dfrac{v}{1 - \sqrt{v}} - v$

i.e., $x\,\dfrac{dv}{dx} = \dfrac{v - v + v\sqrt{v}}{1 - \sqrt{v}}$ or $x\,\dfrac{dv}{dx} = \dfrac{v\sqrt{v}}{1 - \sqrt{v}}$

$\therefore$ $\dfrac{1 - \sqrt{v}}{v\sqrt{v}}\,dv = \dfrac{dx}{x}$

$\Rightarrow \quad \int v^{-3/2}\, dv - \int \frac{dv}{v} - \int \frac{dx}{x} = c$

ie., $\quad \frac{v^{-1/2}}{-1/2} - \log v - \log x = c$

ie., $\quad \frac{2}{\sqrt{v}} + \log(vx) = -c = k$ *(say)* where $v = y/x$

Thus $\quad 2\sqrt{x/y} + \log y = k,$ is the required solution

5. *Solve* : $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$

>> We have $(y - x) = (y + x)\frac{dy}{dx}$

or $\quad \frac{dy}{dx} = \frac{y - x}{y + x}$

Put $\quad y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes, $v + x\frac{dv}{dx} = \frac{vx - x}{vx + x}$

ie., $\quad v + x\frac{dv}{dx} = \frac{x(v-1)}{x(v+1)}$ or $\quad x\frac{dv}{dx} = \frac{v-1}{v+1} - v$

ie., $\quad x\frac{dv}{dx} = \frac{v - 1 - v^2 - v}{v+1}$ or $\quad x\frac{dv}{dx} = -\frac{(1+v^2)}{(v+1)}$

$\therefore \quad \frac{v+1}{1+v^2}\, dv = -\frac{dx}{x}$

$\Rightarrow \quad \int \frac{v\, dv}{1+v^2} + \int \frac{dv}{1+v^2} + \int \frac{dx}{x} = c$

ie., $\quad \frac{1}{2}\log(1+v^2) + \tan^{-1} v + \log x = c$

ie., $\quad \log\left[\sqrt{1+v^2} \cdot x\right] + \tan^{-1} v = c,$ where $v = y/x$

$\therefore \quad \log\left[\sqrt{1 + (y^2/x^2)} \cdot x\right] + \tan^{-1}(y/x) = c$

Thus $\quad \log\sqrt{x^2 + y^2} + \tan^{-1}(y/x) = c,$ is the required solution

6. *Solve:* $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

>> We have $x\,dy = \left[\,y + \sqrt{x^2+y^2}\,\right]dx$

or $\quad \dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2+y^2}}{x}$ ... (1)

Put $\quad y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$

Now (1) becomes $v + x\,\dfrac{dv}{dx} = \dfrac{v\,x + \sqrt{x^2+v^2x^2}}{x}$

ie., $\quad v + x\,\dfrac{dv}{dx} = \dfrac{x\left[\,v + \sqrt{1+v^2}\,\right]}{x} \quad$ or $\quad x\,\dfrac{dv}{dx} = \sqrt{1+v^2}$

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

$\Rightarrow \quad \displaystyle\int \frac{dv}{\sqrt{1+v^2}} - \int \frac{dx}{x} = c$

ie., $\quad \sinh^{-1} v - \log x = c$, where $v = y/x$.

Thus $\quad \sinh^{-1}(y/x) - \log x = c$, is the required solution

Note $\displaystyle\int \frac{dv}{\sqrt{1+v^2}}$ *is also equal to* $\log\left(v + \sqrt{1+v^2}\right)$

Then the solution becomes $\log\left[\dfrac{y}{x} + \sqrt{1+(y^2/x^2)}\,\right] - \log x = c$

ie., $\quad \log\left[\dfrac{y + \sqrt{x^2+y^2}}{x} \Big/ x\right] = c = \log k$ *(say)*

$\Rightarrow \quad \dfrac{y + \sqrt{x^2+y^2}}{x^2} = k$

Thus $\quad \boldsymbol{y + \sqrt{x^2+y^2} = k\,x^2},\quad$ is the solution.

We have $x\,\dfrac{dy}{dx} = y - \dfrac{y^2}{x}$

or $\quad \dfrac{dy}{dx} = \dfrac{x\,y - y^2}{x^2}$ ... (1)

Put $\quad y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $v + x\dfrac{dv}{dx} = \dfrac{x \cdot vx - v^2x^2}{x^2}$

ie., $\quad v + x\dfrac{dv}{dx} = \dfrac{x^2(v - v^2)}{x^2}$ or $x\dfrac{dv}{dx} = -v^2$

$\therefore \quad \dfrac{dv}{v^2} = -\dfrac{dx}{x}$ by separating the variables

Hence $\displaystyle\int \frac{dv}{v^2} + \int \frac{dx}{x} = c$ *ie.,* $-\dfrac{1}{v} + \log x = c$, where $v = \dfrac{y}{x}$

Thus $\quad \log x - \dfrac{x}{y} = c$, is the required solution.

Remark *The presence of terms involving y/x or x/y in the equation, can at recognized as a possible homogeneous equation.*

We have $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$

Put $\quad \dfrac{y}{x} = v$ or $y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $v + x\dfrac{dv}{dx} = v + \sin v$

ie., $\quad x\dfrac{dv}{dx} = \sin v$ or $\dfrac{dv}{\sin v} = \dfrac{dx}{x}$

Hence $\displaystyle\int \operatorname{cosec} v \, dv - \int \frac{dx}{x} = c$

ie., $\quad \log(\operatorname{cosec} v - \cot v) - \log x = c$

or $\quad \log\left[\dfrac{\operatorname{cosec} v - \cot v}{x}\right] = c = \log k$ *(say)*

$\Rightarrow \quad \operatorname{cosec} v - \cot v = kx$, where $v = y/x$

Thus $\quad \operatorname{cosec}(y/x) - \cot(y/x) = kx$, is the required solution

9. [illegible]

>> We have $\dfrac{dy}{dx} = 1 + \dfrac{y}{x} + \dfrac{y^2}{x^2}$ (1)

Put $\dfrac{y}{x} = v$ or $y = vx$ $\therefore$ $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $v + x\dfrac{dv}{dx} = 1 + v + v^2$

ie., $x\dfrac{dv}{dx} = 1 + v^2$ or $\dfrac{dv}{1+v^2} = \dfrac{dx}{x}$

Hence $\displaystyle\int \frac{dv}{1+v^2} - \int \frac{dx}{x} = c$

ie., $\tan^{-1} v - \log x = c$, where $v = y/x$

Thus $\tan^{-1}(y/x) - \log x = c$, is the required solution

10. Solve $[x \tan(y/x) - y\sec^2(y/x)]\,dx + x\sec^2(y/x)\,d$[illegible]

>> The given equation can be written as,

$$\frac{dy}{dx} = \frac{y\sec^2(y/x) - x\tan(y/x)}{x\sec^2(y/x)} \qquad (1)$$

Put $\dfrac{y}{x} = v$ or $y = vx$ $\therefore$ $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes, $v + x\dfrac{dv}{dx} = \dfrac{vx\sec^2 v - x\tan v}{x\sec^2 v}$

ie., $v + x\dfrac{dv}{dx} = \dfrac{x(v\sec^2 v - \tan v)}{x\sec^2 v}$

ie., $x\dfrac{dv}{dx} = \dfrac{v\sec^2 v - \tan v}{\sec^2 v} - v$

ie., $x\dfrac{dv}{dx} = \dfrac{v\sec^2 v - \tan v - v\sec^2 v}{\sec^2 v}$

ie., $x\dfrac{dv}{dx} = \dfrac{-\tan v}{\sec^2 v}$ or $\dfrac{\sec^2 v}{\tan v}\,dv = -\dfrac{dx}{x}$

Hence $\int \frac{\sec^2 v}{\tan v}\,dv + \int \frac{dx}{x} = c$

ie., $\log(\tan v) + \log x = c$

or $\log(\tan v \cdot x) = c = \log k$ (say),

$\Rightarrow$ $x \tan v = k$ where $v = y/x$

Thus $x\tan(y/x) = k$, is the required solution.

11. *Solve* : $x\frac{dy}{dx} = y(\log y - \log x + 1)$

>> The given equation can be written in the form

$$\frac{dy}{dx} = \frac{y}{x}\left[\log(y/x) + 1\right]$$

Put $y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes $v + x\frac{dv}{dx} = v(\log v + 1)$

ie., $x\frac{dv}{dx} = v\log v$ or $\frac{dv}{v\log v} = \frac{dx}{x}$

Hence $\int \frac{1/v}{\log v}\,dv - \int \frac{dx}{x} = c$

ie., $\log(\log v) - \log x = c = \log k$ (say)

ie., $\log(\log v) = \log k + \log x$

ie., $\log(\log v) = \log(kx)$

$\Rightarrow$ $\log v = kx$ where $v = y/x$

Thus $\log(y/x) = kx$, is the required solution

12. *Solve* : $(x - y\log y + y\log x)\,dx + x(\log y - \log x)\,dy = 0$

>> We have, $[x - y\log(y/x)]\,dx + x\log(y/x)\,dy = 0$

Hence $\frac{dy}{dx} = \frac{y\log(y/x) - x}{x\log(y/x)}$

This is a homogeneous equation.

Put $y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence (1) becomes,

$$v + x\frac{dv}{dx} = \frac{v\,x\log v - x}{x\log v}$$

ie., $$x\frac{dv}{dx} = \frac{v\log v - 1}{\log v} - v$$

ie., $$x\frac{dv}{dx} = \frac{v\log v - 1 - v\log v}{\log v}$$

ie., $$x\frac{dv}{dx} = \frac{-1}{\log v}$$

or $$\log v\,dv = -\frac{dx}{x}$$

$\Rightarrow$ $$\int \log v\cdot 1\,dv + \int\frac{dx}{x} = c$$

ie., $$\log v\cdot v - \int v\cdot\frac{1}{v}\,dv + \log x = c$$

ie., $$v\log v - v + \log x = c$$

ie., $$v(\log v - 1) + \log x = c \quad \text{where } v = y/x$$

Thus $(y/x)\,[\log(y/x) - 1] + \log x = c$, is the required solution

13. *Solve :* $(x^2 + 7xy + 16y^2)\,dx + x^2\,dy = 0$ *given that* $y(1) = 1$

>> We have $x^2\,dy = -(x^2 + 7xy + 16y^2)\,dx$

or $$\frac{dy}{dx} = -\frac{(x^2 + 7xy + 16y^2)}{x^2} \quad \ldots(1)$$

Put $y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $$v + x\frac{dv}{dx} = -\frac{(x^2 + 7vx^2 + 16v^2x^2)}{x^2}$$

ie., $$v + x\frac{dv}{dx} = -(1 + 7v + 16v^2)$$

ie., $$x\frac{dv}{dx} = -(16v^2 + 8v + 1)$$

or $$\frac{dv}{16v^2 + 8v + 1} = -\frac{dx}{x}$$ by separating the variables.

$\Rightarrow$ $$\int\frac{dv}{(4v+1)^2} + \int\frac{dx}{x} = c$$

ie., $$\frac{(4v+1)^{-1}}{-1\cdot 4} + \log x = c$$

ie., $$-\frac{1}{4}\left(\frac{1}{4v+1}\right) + \log x = c, \quad \text{where } v = y/x$$

$\therefore$ $$-\frac{1}{4}\left(\frac{x}{4y+x}\right) + \log x = c$$

This is the general solution of the equation.

We have the condition $y(1) = 1$ That is $y = 1$ when $x = 1$

Hence (2) becomes $-\frac{1}{20} + 0 = c \quad \therefore \quad c = -\frac{1}{20}$

Now (2) becomes $-\frac{1}{4}\left(\frac{x}{4y+x}\right) + \log x = -\frac{1}{20}$

ie., $5x - 20\log x\,(4y+x) = 4y + x$

ie., $4x - 4y - 20\log x\,(4y+x) = 0$

Thus $x - y - 5\log x\,(x+4y) = 0$ is the required particular solution

14. Solve : $\frac{dy}{dx} = \frac{x+2y}{2x+y}$

>> Put $y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now the given equation becomes

$$v + x\frac{dv}{dx} = \frac{x + 2vx}{2x + vx}$$

$$v + x\frac{dv}{dx} = \frac{1+2v}{2+v}$$

or $$x\frac{dv}{dx} = \frac{1+2v}{2+v} - v$$

ie., $$x\frac{dv}{dx} = \frac{1-v^2}{2+v} \quad \text{or} \quad \frac{2+v}{1-v^2}\,dv = \frac{dx}{x}$$

$$\int \frac{2+v}{1-v^2}\,dv - \int \frac{dx}{x} = c$$

$$2\int \frac{dv}{1-v^2} + \int \frac{v\,dv}{1-v^2} - \log x = c$$

$$2\cdot\frac{1}{2}\log\left(\frac{1+v}{1-v}\right)-\frac{1}{2}\log(1-v^2)-\log x = c$$

$$\log\left(\frac{1+v}{1-v}\right)-\log\left[\sqrt{1-v^2}\cdot x\right] = c$$

$$\log\left[\frac{1+v}{(1-v)\sqrt{1-v^2}\,x}\right] = \log k \ (say)$$

$\Rightarrow$ $$1+v = kx(1-v)\sqrt{1-v^2}$$

ie., $$1+v = kx(1-v)\sqrt{1-v}\sqrt{1+v}$$

ie., $$\sqrt{1+v} = kx(1-v)\sqrt{1-v} \quad \text{where } v = y/x$$

ie., $$\sqrt{(x+y)/x} = kx\left(\frac{x-y}{x}\right)\sqrt{(x-y)/x}$$

ie., $$\sqrt{x+y} = k(x-y)^{3/2}$$

Thus $x+y = k^2(x-y)^3$, is the required solution

(As we observe terms with x/y, we need to express the equation relating to dx/dy and the terms are homogeneous functions of degree 0)

>> We have $(1+e^{x/y})\,dx = e^{x/y}\left(\frac{x}{y}-1\right)dy$

or $$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{(1+e^{x/y})} \qquad \ldots(1)$$

Put $\frac{x}{y} = v$ or $x = vy$ $\therefore$ $\frac{dx}{dy} = v+y\frac{dv}{dy}$

Now (1) becomes, $v+y\frac{dv}{dy} = \frac{e^v(v-1)}{(1+e^v)}$

ie., $y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}-v$ or $y\frac{dv}{dy} = \frac{e^v v-e^v-v-e^v v}{1+e^v}$

ie., $y\frac{dv}{dy} = -\frac{(e^v+v)}{(1+e^v)}$ or $\frac{(1+e^v)\,dv}{e^v+v} = -\frac{dy}{y}$

Hence $\int\frac{(1+e^v)\,dv}{e^v+v}+\int\frac{dy}{y} = c$

ie., $\log(e^v+v)+\log y = c$

or $\quad \log\left[(e^v + v)\, y \right] = \log k \ \ (say)$

$\Rightarrow \quad (e^v + v)\, y = k, \quad \text{where } v = x/y$

Thus $\quad y\, e^{x/y} + x = k, \quad$ is the required solution.

16. Solve: $[x + y \cot(x/y)]\, dy - y\, dx = 0$

>> We have $[x + y \cot(x/y)]\, dy = y\, dx$

$$\therefore \quad \frac{dx}{dy} = \frac{x + y \cot(x/y)}{y}$$

Put $\quad \dfrac{x}{y} = v \ \text{ or } \ x = v\,y \quad \therefore \quad \dfrac{dx}{dy} = v + y\,\dfrac{dv}{dy}$

Now (1) becomes, $\quad v + y\,\dfrac{dv}{dy} = \dfrac{v\,y + y \cot v}{y}$

ie., $\quad v + y\,\dfrac{dv}{dy} = \dfrac{y\,(v + \cot v)}{y} \quad \text{or} \quad y\,\dfrac{dv}{dy} = \cot v$

$\therefore \quad \dfrac{dv}{\cot v} = \dfrac{dy}{y}$ by separating the variables.

$$\Rightarrow \quad \int \tan v\, dv - \int \frac{dy}{y} = c$$

ie., $\quad \log(\sec v) - \log y = c \quad \text{or} \quad \log\left[\dfrac{\sec v}{y}\right] = \log k \ \ (say)$

Hence we have $\sec v = k\,y, \quad \text{where } v = x/y$

Thus $\quad \sec(x/y) = k\,y, \quad$ is the required solution.

17. Solve: $y\, dy + \sin^2(x/y)\,[x\, dy - y\, dx] = 0$

>> We have $[y + x \sin^2(x/y)]\, dy = y \sin^2(x/y)\, dx$

$$\therefore \quad \frac{dx}{dy} = \frac{y + x \sin^2(x/y)}{y \sin^2(x/y)}$$

Put $\quad \dfrac{x}{y} = v \ \text{ or } \ x = v\,y \quad \therefore \quad \dfrac{dx}{dy} = v + y\,\dfrac{dv}{dy}$

Now (1) becomes, $\quad v + y\,\dfrac{dv}{dy} = \dfrac{y + v\,y \sin^2 v}{y \sin^2 v}$

ie., $\quad v + y\,\dfrac{dv}{dy} = \dfrac{y\,(1 + v \sin^2 v)}{y \sin^2 v} \quad \text{or} \quad y\,\dfrac{dv}{dy} = \dfrac{1 + v \sin^2 v}{\sin^2 v} - v$

ie., $y\,\dfrac{dv}{dy} = \dfrac{1 + v\sin^2 v - v\sin^2 v}{\sin^2 v}$ or $y\,\dfrac{dv}{dy} = \dfrac{1}{\sin^2 v}$

$\therefore$ $\sin^2 v\,dv = \dfrac{dy}{y}$ by separating the variables.

$$\int \sin^2 v\,dv - \int \frac{dy}{y} = c \quad \text{or} \quad \int \frac{1-\cos 2v}{2}\,dv - \log y = c$$

ie., $\dfrac{1}{2}\displaystyle\int dv - \dfrac{1}{2}\int \cos 2v\,dv - \log y = c$

ie., $\dfrac{v}{2} - \dfrac{\sin 2v}{4} - \log y = c$, where $v = x/y$

Thus $\dfrac{x}{2y} - \dfrac{1}{4}\sin(2x/y) - \log y = c$ is the required solution

6.33 Equations reducible to the homogeneous form

Consider the differential equation in the form :

$$(ax + by + c)\,dx \pm (a'x + b'y + c')\,dy = 0$$

We first express the equation in respect of $\dfrac{dy}{dx}$ and the procedure is narrated by taking

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \quad \text{where } \frac{a}{a'} \neq \frac{b}{b'} \qquad \ldots(1)$$

This condition implies that there are no common factors for the x and y terms in the numerator as well as in the denominator.

Put $x = X + h$, $y = Y + k$ where h and k are constants to be chosen appropriately later

Now $\dfrac{dy}{dx} = \dfrac{dy}{dY}\,\dfrac{dY}{dX}\,\dfrac{dX}{dx}$

$= 1 \cdot \dfrac{dY}{dX} \cdot 1$ Hence $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

As a consequence of these (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'}$$

ie., $$\frac{dY}{dX} = \frac{(aX + bY) + (ah + bk + c)}{(a'X + b'Y) + (a'h + b'k + c')} \qquad \ldots(2)$$

Now, let us choose h and k such that :

$$a\,h + b\,k + c = 0 \quad \text{and} \quad a'\,h + b'\,k + c' = 0$$

Solving these equations we get the value for h and k

Thus (2) now assumes the form

$$\frac{dY}{dX} = \frac{aX + bY}{a'\,X + b'\,Y}$$

It is evident that (3) is a homogeneous equation in the variables X and [illegible] equation can be solved by putting $Y = VX$ as discussed already. Fin[illegible] substitute for X and Y where $X = x - h,\ \ Y = y - k.$

Remark : If $\dfrac{a}{a'} = \dfrac{b}{b'} = k$ (say) *then* $a = a'\,k,\ b = b'\,k.$

Hence (1) *becomes* $\dfrac{dy}{dx} = \dfrac{a'\,k\,x + b'\,k\,y + c}{a'\,x + b'\,y + c'}$

or $$\frac{dy}{dx} = \frac{k\,(a'\,x + b'\,y) + c}{(a'\,x + b'\,y) + c'}$$

We can solve by putting $t = a'\,x + b'\,y$ *(Reducible to variables separable)*

WORKED PR[illegible]

18. [illegible]

>> We have $(4x + y - 2)\,dy = -(x - 4y - 9)\,dx$

$\therefore$ $$\frac{dy}{dx} = \frac{-x + 4y + 9}{4x + y - 2}$$

Put $x = X + h$ and $y = Y + k$ where h and k are constants to be chosen [illegible] later.

Now $$\frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \qquad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes,

$$\frac{dY}{dX} = \frac{-(X + h) + 4(Y + k) + 9}{4(X + h) + (Y + k) - 2}$$

ie., $$\frac{dY}{dX} = \frac{(-X + 4Y) + (-h + 4k + 9)}{(4X + Y) + (4h + k - 2)}$$

Let us choose h and k such that

$$-h+4k+9=0 \text{ and } 4h+k-2=0$$

Solving these equations we get, $h=1$ and $k=-2$

Thus (2) becomes $\dfrac{dY}{dX}=\dfrac{-X+4Y}{4X+Y}$ ... (3)

Put $\quad Y=VX \Rightarrow \dfrac{dY}{dX}=V+X\dfrac{dV}{dx}$

Now (3) becomes, $V+X\dfrac{dV}{dX}=\dfrac{-X+4VX}{4X+VX}$

ie., $\quad V+X\dfrac{dV}{dX}=\dfrac{X(-1+4V)}{X(4+V)}$ or $X\dfrac{dV}{dX}=\dfrac{-1+4V}{4+V}-V$

ie., $\quad X\dfrac{dV}{dX}=\dfrac{-1+4V-4V-V^2}{4+V}$ ie., $X\dfrac{dV}{dX}=\dfrac{-(1+V^2)}{(4+V)}$

$\dfrac{(4+V)\,dV}{1+V^2}=-\dfrac{dX}{X}$ by separating the variables.

$\Rightarrow \quad 4\displaystyle\int\frac{dV}{1+V^2}+\int\frac{V\,dV}{1+V^2}+\int\frac{dX}{X}=c$

ie., $\quad 4\tan^{-1}V+\dfrac{1}{2}\log(1+V^2)+\log X=c$

ie., $\quad 8\tan^{-1}V+\log(1+V^2)+2\log X=2c$

ie., $\quad 8\tan^{-1}V+\log\left[(1+V^2)X^2\right]=2c$, where $V=Y/X$

$\therefore \quad 8\tan^{-1}(Y/X)+\log(X^2+Y^2)=2c=k$ (say)

But $\quad X=x-h=x-1$ and $Y=y-k=y+2$

Thus $\quad 8\tan^{-1}\left(\dfrac{y+2}{x-1}\right)+\log\left[(x-1)^2+(y+2)^2\right]=k$, is the required solution

>> We have $\dfrac{dy}{dx}=\dfrac{x+y-1}{x-y+1}$ ... (1)

Put $x=X+h$ and $y=Y+k$, where h and k are constants to be chosen suitably later

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \quad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes,

$$\frac{dY}{dX} = \frac{(X+h)+(Y+k)-1}{(X+h)-(Y+k)+1}$$

ie., $$\frac{dY}{dX} = \frac{(X+Y)+(h+k-1)}{(X-Y)+(h-k+1)} \qquad \ldots (2)$$

Let us choose h and k such that

$$h+k-1 = 0 \text{ and } h-k+1 = 0$$

Solving these equations we get, $h = 0,\ k = 1$.

Thus (2) becomes, $\dfrac{dY}{dX} = \dfrac{X+Y}{X-Y}$ $\qquad \ldots (3)$

Put $Y = VX \quad \therefore \quad \dfrac{dY}{dX} = V + X\,\dfrac{dV}{dX}$

Now (3) becomes, $V + X\,\dfrac{dV}{dX} = \dfrac{X+VX}{X-VX}$

ie., $V + X\,\dfrac{dV}{dX} = \dfrac{X(1+V)}{X(1-V)}$ or $X\,\dfrac{dV}{dX} = \dfrac{1+V}{1-V} - V$

ie., $X\,\dfrac{dV}{dX} = \dfrac{1+V-V+V^2}{1-V}$ *ie.,* $X\,\dfrac{dV}{dX} = \dfrac{1+V^2}{1-V}$

$\therefore \quad \dfrac{1-V}{1+V^2}\,dV = \dfrac{dX}{X}$ by separating the variables.

$\Rightarrow \quad \displaystyle\int \frac{dV}{1+V^2} - \int \frac{V\,dV}{1+V^2} - \int \frac{dX}{X} = c$

ie., $\tan^{-1} V - \dfrac{1}{2}\log(1+V^2) - \log X = c$

ie., $2\tan^{-1} V - \log(1+V^2) - 2\log X = 2c$

ie., $2\tan^{-1} V - \log\left[(1+V^2)X^2\right] = 2c$, where $V = Y/X$

ie., $2\tan^{-1}(Y/X) - \log\left[X^2+Y^2\right] = 2c = k$ (say)

But $X = x-h = x$ and $Y = y-k = y-1$

Thus $2\tan^{-1}\left(\dfrac{y-1}{x}\right) - \log\left[x^2+(y-1)^2\right] = k$ is the required solution.

20. Solve $(3y-7x+7)\,dx+(7y-3x+3)\,dy=0$

>> We have $(7y-3x+3)\,dy = (7x-3y-7)\,dx$

$$\frac{dy}{dx} = \frac{7x-3y-7}{-3x+7y+3} \qquad \ldots(1)$$

Put $x = X+h$ and $y = Y+k$, where h and k are constants to be chosen suitably later.

$$\frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$= 1\cdot\frac{dY}{dX}\cdot 1 \qquad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes,

$$\frac{dY}{dX} = \frac{7(X+h)-3(Y+k)-7}{-3(X+h)+7(Y+k)+3}$$

ie., $$\frac{dY}{dX} = \frac{(7X-3Y)+(7h-3k-7)}{(-3X+7Y)+(-3h+7k+3)} \qquad \ldots(2)$$

Let us choose h and k such that,

$$7h-3k-7 = 0 \text{ and } -3h+7k+3 = 0$$

Solving these equations we get, $h = 1,\ k = 0$.

Thus (2) becomes, $$\frac{dY}{dX} = \frac{7X-3Y}{-3X+7Y} \qquad \ldots(3)$$

Put $\quad Y = VX,\ \therefore\ \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

Now (3) becomes $$V + X\frac{dV}{dX} = \frac{7X-3VX}{-3X+7VX}$$

ie., $$V + X\frac{dV}{dX} = \frac{X(7-3V)}{X(-3+7V)}$$

ie., $$X\frac{dV}{dX} = \frac{7-3V}{-3+7V} - V \quad \text{or} \quad X\frac{dV}{dX} = \frac{7-7V^2}{-3+7V}$$

$$\frac{-3+7V}{1-V^2}\,dV = 7\,\frac{dX}{X} \quad \text{by separating the variables}$$

$$\Rightarrow \qquad -3\int\frac{dV}{1-V^2} + 7\int\frac{V\,dV}{1-V^2} - 7\int\frac{dX}{X} = c$$

Using $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$ for the first term we have,

$$-\frac{3}{2} \log\left(\frac{1+V}{1-V}\right) - \frac{7}{2} \log(1-V^2) - 7 \log X = c$$

or $$3 \log\left(\frac{1+V}{1-V}\right) + 7 \log(1-V^2) + 14 \log X = -2c$$

or $$\log\left[\left(\frac{1+V}{1-V}\right)^3 (1-V^2)^7 X^{14}\right] = \log c_1 \quad (say)$$

$\Rightarrow$ $$\frac{(1+V)^3}{(1-V)^3}(1-V)^7(1+V)^7 X^{14} = c_1$$

or $$(1+V)^{10}(1-V)^4 X^{14} = c_1$$

Taking square root we have,

$$(1+V)^5(1-V)^2 X^7 = \sqrt{c_1}, \quad \text{where } V = Y/X$$

ie., $$\left(\frac{X+Y}{X}\right)^5 \left(\frac{X-Y}{X}\right)^2 X^7 = k \ (say) \text{ where } k = \sqrt{c_1}$$

ie., $$(X+Y)^5(X-Y)^2 = k$$

But $X = x - h = x - 1$ and $Y = y - k = y$

Thus $(x+y-1)^5(x-y-1)^2 = k$, is the required solution

'I

>> We have $$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \qquad (1$$

Put $x = X + h$ and $y = Y + k$, where h and k are constants to be chosen suitabl later.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \quad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes, $$\frac{dY}{dX} = \frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3}$$

ie., $$\frac{dY}{dX} = \frac{(X+2Y)+(h+2k-3)}{(2X+Y)+(2h+k-3)} \qquad \ldots (2)$$

Let us choose h and k such that,

$$h+2k-3 = 0 \text{ and } 2h+k-3 = 0$$

Solving these equations we get $h = 1$ and $k = 1$

Thus (2) becomes $$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} \qquad \ldots (3)$$

Put $Y = VX \quad \therefore \quad \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

Now (3) becomes $V + X\dfrac{dV}{dX} = \dfrac{X+2VX}{2X+VX}$

ie., $$V + X\frac{dV}{dX} = \frac{X(1+2V)}{X(2+V)}$$

or $$X\frac{dV}{dX} = \frac{1+2V}{2+V} - V \text{ or } X\frac{dV}{dX} = \frac{1-V^2}{2+V}$$

$\therefore$ $\dfrac{2+V}{1-V^2}\,dV = \dfrac{dX}{X}$ by separating the variables

$\Rightarrow$ $$2\int\frac{dV}{1-V^2} + \int\frac{V\,dV}{1-V^2} - \int\frac{dX}{X} = c$$

ie., $$2\cdot\frac{1}{2}\log\left(\frac{1+V}{1-V}\right) - \frac{1}{2}\log(1-V^2) - \log X = c$$

ie., $$2\log\left(\frac{1+V}{1-V}\right) - \log(1-V^2) - 2\log X = 2c$$

ie., $$\log\left(\frac{1+V}{1-V}\right)^2 - \log\left[(1-V^2)X^2\right] = 2c$$

or $$\log\left[\frac{(1+V)^2}{(1-V)^2(1-V^2)X^2}\right] = \log k \text{ (say)}$$

$\Rightarrow$ $$\frac{(1+V)}{(1-V)^3X^2} = k \quad \text{or} \quad \frac{(X+Y)}{(X-Y)^3} = k \text{ since } V = \frac{Y}{X}$$

But $X = x-h = x-1$ and $Y = y-k = y-1$

Thus $(x+y-2) = k(x-y)^3$, *is the required solution*

>> We have $\dfrac{dy}{dx} = \dfrac{y-x+5}{y+x+3}$

Put $x = X+h$ and $y = Y+k$, where h and k are constants to be chosen late

$$\frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \quad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes

$$\frac{dY}{dX} = \frac{(Y+k)-(X+h)+5}{(Y+k)+(X+h)+3}$$

$$\frac{dY}{dX} = \frac{(Y-X)+(k-h+5)}{(Y+X)+(k+h+3)}$$

Let us choose h and k such that

$$k-h+5 = 0 \text{ and } k+h+3 = 0$$

By solving these equations we get $h = 1$ and $k = -4$.

Thus (2) becomes $\dfrac{dY}{dX} = \dfrac{Y-X}{Y+X}$

Put $Y = VX \ \therefore\ \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

Now (3) becomes,

$$V + X\frac{dV}{dX} = \frac{VX-X}{VX+X}$$

or $\quad X\dfrac{dV}{dX} = \dfrac{V-1}{V+1} - V$ or $X\dfrac{dV}{dX} = -\dfrac{(1+V^2)}{V+1}$

$\therefore \quad \dfrac{V+1}{1+V^2}\,dV = \dfrac{-dX}{X}$ by separating the variables.

$\Rightarrow \quad \displaystyle\int \frac{V+1}{1+V^2}\,dV + \int \frac{dX}{X} = c$

ie., $\quad \displaystyle\int \frac{VdV}{1+V^2} + \int \frac{dV}{1+V^2} + \log X = c$

ie., $\quad \dfrac{1}{2}\log(1+V^2) + \tan^{-1} V + \log X = c$

ie., $\log [X\sqrt{1+V^2}] + \tan^{-1} V = c$ where $V = Y/X$

ie., $\log [X\sqrt{1+(Y^2/X^2)}] + \tan^{-1}(Y/X) = c$

ie., $\log \sqrt{X^2+Y^2} + \tan^{-1}(Y/X) = c$

But $X = x-h = x-1$ and $Y = y-k = y+4$

Thus $\log \sqrt{(x-1)^2+(y+4)^2} + \tan^{-1}[(y+4)/(x-1)] = c$,
is the required solution.

23. *Find the particular solution of the e*
$(x-1)\,dx - (3x-2y-5)\,dy = 0$

>> We have $(x-1)\,dx = (3x-2y-5)\,dy$

$$\frac{dy}{dx} = \frac{x-1}{3x-2y-5} \qquad \ldots (1)$$

Put $x = X+h$ and $y = Y+k$ where h and k are constants to be chosen later.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx}$$

$$= 1\cdot\frac{dY}{dX}\cdot 1 \qquad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes $\dfrac{dY}{dX} = \dfrac{(X+h)-1}{3(X+h)-2(Y+k)-5}$

ie., $$\frac{dY}{dX} = \frac{X+(h-1)}{(3X-2Y)+(3h-2k-5)} \qquad \ldots (2)$$

Let us choose h and k such that

$$h-1 = 0 \text{ and } 3h-2k-5 = 0$$

These equations will give us $h = 1$, $k = -1$ on solving

Thus (2) becomes, $\dfrac{dY}{dX} = \dfrac{X}{3X-2Y} \qquad (3)$

Put $Y = VX \quad \therefore \quad \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

Now (3) becomes, $V + X\dfrac{dV}{dX} = \dfrac{X}{3X-2VX}$

ie., $$V + X\frac{dV}{dX} = \frac{X}{X(3-2V)}$$

or $$X\frac{dV}{dX} = \frac{1}{3-2V} - V$$

ie., $$X\frac{dV}{dX} = \frac{1-3V+2V^2}{3-2V}$$

$$\therefore \quad \frac{3-2V}{1-3V+2V^2}\,dV = \frac{dX}{X}$$ by separating the variables.

$$\Rightarrow \quad \int\frac{3-2V}{2V^2-3V+1} - \int\frac{dX}{X} = c$$

We have $2V^2-3V+1 = (2V-1)(V-1)$ by factorization

Now, let $$\frac{3-2V}{(2V-1)(V-1)} = \frac{A}{2V-1}+\frac{B}{V-1}$$

or $\quad 3-2V = A(V-1)+B(2V-1)$

Put $\quad V = 1 \quad : \; 1 = B(1) \qquad \therefore\; B = 1$

Put $\quad V = 1/2 \quad : \; 2 = A(-1/2) \qquad \therefore\; A = -4$

Now $$\int\frac{(3-2V)\,dV}{(2V-1)(V-1)} = -4\int\frac{dV}{2V-1}+\int\frac{dV}{V-1}$$

ie., $$\int\frac{(3-2V)\,dV}{2V^2-3V+1} = -2\log(2V-1)+\log(V-1)$$

Using (5) in (4) we have

$$-2\log(2V-1)+\log(V-1)-\log X = c$$

or $$\log\left[\frac{V-1}{(2V-1)^2X}\right] = \log k \quad (say)$$

$$\Rightarrow \quad \frac{V-1}{(2V-1)^2X} = k \text{ where } V = \frac{Y}{X}$$

$$\therefore \quad \frac{Y-X}{(2Y-X)^2} = k \text{ or } (Y-X) = k(2Y-X)^2$$

But $\quad X = x-h = x-1$ and $Y = y-k = y+1$

$\therefore\; (y-x+2) = k(2y-x+3)^2$ is the general solution of the equation.

Now consider $y(1) = 2$. That is $y = 2$ when $x = 1$.

The general solution becomes $3 = k(36)$ or $k = 1/12$

Thus $\quad 12(y-x+2) = (2y-x+3)^2$ is the required particular solution.

24. Solve $(2x+y-1)\,dy = (x-2y+5)\,dx$

>> $$\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1} \quad \ldots (1)$$

Put $x = X+h$ and $y = Y+k$, where h and k are constants to be chosen later

$$\frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$\frac{dy}{dx} = 1\cdot\frac{dY}{dX}\cdot 1 = \frac{dY}{dX}$$

Hence (1) becomes,

$$\frac{dY}{dX} = \frac{(X+h)-2(Y+k)+5}{2(X+h)+(Y+k)-1}$$

ie., $$\frac{dY}{dX} = \frac{(X-2Y)+(h-2k+5)}{(2X+Y)+(2h+k-1)} \quad \ldots (2)$$

Let us choose h and k such that,

$$h-2k+5 = 0 \text{ and } 2h+k-1 = 0$$

Solving these equations we get

$$h = -3/5 \text{ and } k = 11/5$$

Now (2) becomes,

$$\frac{dY}{dX} = \frac{X-2Y}{2X+Y} \quad \ldots (3)$$

Put $Y = VX$

$$\therefore \frac{dY}{dX} = V + X\frac{dV}{dX}$$

Hence (3) becomes,

$$V + X\frac{dV}{dX} = \frac{X(1-2V)}{X(2+V)} \quad \text{or} \quad X\frac{dV}{dX} = \frac{1-2V}{2+V} - V$$

ie., $$X\frac{dV}{dX} = \frac{1-4V-V^2}{2+V} \quad \text{or} \quad \frac{2+V}{1-4V-V^2}\,dV = \frac{dX}{X},$$ by separating the variables

$$\Rightarrow \int \frac{2+V}{1-4V-V^2}\,dV - \frac{dX}{X} = c$$

ie., $$-\frac{1}{2}\log(1-4V-V^2) - \log X = c$$

or $\quad \log [(1-4V-V^2) X^2] = -2c = \log c_1$ (*say*)

$\Rightarrow \quad (1-4V-V^2) X^2 = c_1$, where $V = \dfrac{Y}{X}$

i.e., $\quad \left(1-\dfrac{4Y}{X}-\dfrac{Y^2}{X^2}\right) X^2 = c_1 \quad$ or $\quad X^2-4XY-Y^2 = c_1 \quad \ldots (4)$

But $X = x-h = x+(3/5)$ and $Y = y-k = y-(11/5)$

Using these in (4) and simplifying we obtain the required solution in the form

$$\boldsymbol{x^2-4xy-y^2+10x+2y = k}$$

Solve the following equations.

1. $(xy+y^2)\,dx - x^2\,dy = 0$
2. $(x^3+y^3)\,dx = xy(x+y)\,dy$
3. $y\,dx - x\,dy = \sqrt{y^2-x^2}\,dy$
4. $(2x+y)^2 = xy\dfrac{dy}{dx}$
5. $x\,dy = [y - x\cos^2(y/x)]\,dx$
6. $\dfrac{dy}{dx} + \dfrac{x^3+3xy^2}{y^3+3x^2y} = 0$
7. $x[dx + \sin(y/x)\,dy] = y\sin(y/x)\,dx$
8. $[x\,dy - y\,dx]\,y\sin(y/x) = [x\,dy + y\,dx]\,x\cos(y/x)$
9. $(x + y\log x - y\log y)\,dx - x(\log x - \log y)\,dy = 0$
10. $e^{x/y}[y\,dx - x\,dy] = y^2\,dy$
11. $(x^2 - 4xy - 2y^2)\,dx - (2x^2 + 4xy - y^2)\,dy = 0$, $y(0) = 2$
12. $(y-x-4)\,dx = (y+x-2)\,dy$
13. $(y-2)\,dx + (y-x+1)\,dy = 0$; $y(5) = 3$
14. $(3x-y-1)\,dy + 2(x-y)\,dx = 0$
15. $\dfrac{dy}{dx} + \dfrac{2x+3y}{y+2} = 0$

ANSWERS

1. $x/y + \log x = c$
2. $y/x + \log(y - x) = cx$
3. $\sin^{-1}(x/y) - \log y = c$
4. $\log(1 + y/x) + 4(\log x - 1/x) = c$
5. $\log x + \tan(y/x) = c$
6. $x^4 + 6x^2y^2 + y^4 = c$
7. $\log x - \cos(y/x) = c$
8. $xy\cos(y/x) = c$
9. $\log x + (y/x)[\log(y/x) - 1] = c$
10. $e^{x/y} - y = c$
11. $x^3 - 6x^2y - 6xy^2 + y^3 = 8$
12. $\log[(x+1)^2 + (y-3)^2] + 2\tan^{-1}[(y-3)/(x+1)] = c$
13. $\log(y-2) + \{(x-3) \quad (y-2)\} = 2$
14. $(x + y - 1)^4 = c(4x - 2y - 1)$
15. $(2x + y - 4)^2 = c(x + y - 1)$

We introduce this topic with an illustration.

Consider a function $f(x, y) = x^2 + xy + y^2 + x + y = c$ where c is an arbitrary constant.

Let us take the differential of this function.

i.e., $\quad df = 2x\,dx + x\,dy + y\,dx + 2y\,dy + dx + dy = 0$

i.e., $\quad (2x + y + 1)\,dx + (x + 2y + 1)\,dy = 0$

Obviously we can say that the solution of this differential equation is $f(x, y) = c$ where $f(x, y)$ is the function we started with.

In other words, if $M(x, y) = 2x + y + 1$ and $N(x, y) = x + 2y + 1$ then

$$d[f(x, y)] = M(x, y)\,dx + N(x, y)\,dy$$

the solution of the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is equivalent to the solution of the equation

$$d[f(x, y)] = 0 \text{ which is } f(x, y) = c, \text{ on integration.}$$

In other words, if we are able to identify that in a given differential equation :

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

$$M(x, y)\,dx + N(x, y)\,dy = d\,[f(x, y)]$$

then we can simply conclude that the solution of the equation is $f(x, y) = c$

Thus we say that $M(x, y)\,dx + N(x, y)\,dy = 0$ is an exact differential equation *exists a function $f(x, y)$ such that*

$$df = M(x, y)\,dx + N(x, y)\,dy.$$

How to identify that the given equation is exact ? If so what is its solution ? The questions are answered in the following **theorem**

Statement The necessary and the suffecient condition for the differential eq $M(x, y)\,dx + N(x, y)\,dy = 0$ to be an exact equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Further **the solution of the exact equation is** given by

$$\int M\,dx + \int N(y)\,dy = c$$

where, in the first term we integrate $M(x, y)$ *w.r.t* x keeping y fixe $N(y)$ indicate the terms in N without x (*not containing* x)

Important Note *This method is a very easy method for solving a differential equatio Some of the equations of the types reducible to the variables separable form, homogene and reducible to the homogeneous form might be an exact equation. Therefore it is very much advisible that if an equation without the involvement of y/x or x/y terms is homogeneous then exactness is checked before proceeding to solve by putting $y = vx$ $x = vy$. However in the case of terms involving y/x or x/y homogeneous equation solving procedure itself is easier. This method should be tried in the case of problems wh are of the type*

$$(ax + by + c)\,dx \pm (a'x + b'y + c')\,dy = 0.$$

First few problems in this method to follow belong to this category

WORKED PROBLEMS

25. *Solve* $(2x + y + 1)\,dx + (x + 2y + 1)\,dy = 0$

Note *The given equation can be written in the form* $\dfrac{dy}{dx} = -\dfrac{2x + y + 1}{x + 2y + 1}$ *and we have a discussed the method of solving this equation by putting $x = X + h$ and $y = Y + k$ one ventures to do so, the condition for an exact d.e. be checked by just scribbl*

\>> Let $M = 2x + y + 1$ and $N = x + 2y + 1$

$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact

The solution is $\int M\,dx + \int N(y)\,dy = c$, where $N(y)$ denotes terms in N not containing x.

ie., $\int (2x+y+1)\,dx + \int (2y+1)\,dy = c$

ie., $x^2 + xy + x + y^2 + y = c$

Thus $x^2 + xy + y^2 + x + y = c$, is the required solution

Remark *This example is the illustrative example which we took to introduce the concept of an exact d.e. and the solution that we have obtained is same as $f(x, y)$ in the illustration. The method is so simple as one can imagine the amount of work involved had it been done by reducing it to a homogeneous equation.*

26. *Solve :* $(y^3 - 3x^2y)\,dx - (x^3 - 3xy^2)\,dy = 0$

Note : *Observe that this is a homogeneous equation*

>> Let $M = y^3 - 3x^2y$ and $N = -x^3 + 3xy^2$

$$\frac{\partial M}{\partial y} = 3y^2 - 3x^2 \text{ and } \frac{\partial N}{\partial x} = -3x^2 + 3y^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (y^3 - 3x^2y)\,dx + \int 0\,dy = c$

(Observe that there is no term in N which do not contain x)

$\therefore \quad y^3 \cdot x - x^3y = c$

Thus $xy^3 - x^3y = c$, is the required solution.

Remark *Compare the working and the answer of this Problem with Problem 3 solved by putting $y = vx$.*

27. *Solve :* $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

[Though it is evident that the equation is a homogeneous one before solving by putting $y = vx$ we should check for exactness]

>> Let $M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2 y - 6xy^2 \text{ and } \frac{\partial N}{\partial x} = 6x^2 y - 6xy^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int \left(5x^4 + 3x^2 y^2 - 2xy^3\right) dx + \int -5y^4\,dy = c$

Thus $x^5 + x^3 y^2 - x^2 y^3 - y^5 = c$, is the required solution

Note *The problem can be solved by writing the equation in the form*

$$\frac{dy}{dx} = -\frac{(x+3y)-4}{3(x+3y)-2} \text{ and then by putting } x + 3y = t$$

>> The given equation is equivalent to the form

$$(x + 3y - 4)\,dx + (3x + 9y - 2)\,dy = 0$$

Let $M = x + 3y - 4$ and $N = 3x + 9y - 2$

$\therefore$ $\frac{\partial M}{\partial y} = 3$ and $\frac{\partial N}{\partial x} = 3$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (x + 3y - 4)\,dx + \int (9y - 2)\,dy = c$

ie., $\frac{x^2}{2} + 3xy - 4x + \frac{9y^2}{2} - 2y = c$

Thus $x^2 + 6xy - 8x + 9y^2 - 4y = 2c$, is the required solution

>> Let $M = y(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\frac{\partial M}{\partial y} = 1 + 1/x - \sin y \text{ and } \frac{\partial N}{\partial x} = 1 + 1/x - \sin y$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int \left[y\,(1+1/x) + \cos y \right] dx + \int 0\; dy = c$

Thus $y\,(x + \log x) + x \cos y = c$, is the required solution

>> Let $M = \cos x \tan y + \cos(x+y)$, $N = \sin x \sec^2 y + \cos(x+y)$

$\dfrac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y)$, $\dfrac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int [\cos x \tan y + \cos(x+y)]\,dx + \int 0\; dy = c$

Thus $\sin x \tan y + \sin(x+y) = c$, is the required solution

>> The given equation is put in the form,

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0.$$

Let $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$

$\dfrac{\partial M}{\partial y} = \cos x + \cos y + 1$ and $\dfrac{\partial N}{\partial x} = \cos x + \cos y + 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$

Thus $y\sin x + x\sin y + x\,y = c$, is the required solution

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int \left[y\,(1 + 1/x) + \cos y \right] dx + \int 0\ dy = c$

Thus $y\,(x + \log x) + x\cos y = c$, is the required solution

>> Let $M = \cos x \tan y + \cos(x+y)$, $N = \sin x \sec^2 y + \cos(x+y)$

$\therefore \quad \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y)$, $\frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int [\cos x \tan y + \cos(x+y)]\,dx + \int 0\ dy = c$

Thus $\sin x \tan y + \sin(x+y) = c$, is the required solution

>> The given equation is put in the form,

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0.$$

Let $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ and $\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$

Thus $y\sin x + x\sin y + xy = c$, is the required solution

32. *Solve* $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

>> Let $M = 2xy + y - \tan y$ and $N = x^2 - x\tan^2 y + \sec^2 y$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x - \tan^2 y$$

But $\dfrac{\partial M}{\partial y} = 2x + 1 - (1 + \tan^2 y) = 2x - \tan^2 y$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (2xy + y - \tan y)\,dx + \int \sec^2 y\,dy = c$

Thus $x^2 y + xy - x\tan y + \tan y = c$, is the required solution

33. *Solve* $y e^{xy}\,dx + (x e^{xy} + 2y)\,dy = 0$

>> Let $M = y e^{xy}$, $N = x e^{xy} + 2y$

$$\frac{\partial M}{\partial y} = y e^{xy} x + e^{xy} \; ; \; \frac{\partial N}{\partial x} = x e^{xy} y + e^{xy}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Solution is given by $\int M\,dx + \int N(y)\,dy = c$

ie., $\int y e^{xy}\,dx + \int 2y\,dy = c$

ie., $y\dfrac{e^{xy}}{y} + y^2 = c$

Thus $e^{xy} + y^2 = c$, is the required solution

>> Let $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y \; , \; \frac{\partial N}{\partial x} = 2xy\, e^{xy^2} \cdot y^2 + e^{xy^2} \cdot 2y$$

ie., $\dfrac{\partial M}{\partial y} = 2xy^3 e^{xy^2} + 2y e^{xy^2}$ and $\dfrac{\partial N}{\partial x} = 2xy^3 e^{xy^2} + 2y e^{xy^2}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int \left(y^2 e^{xy^2} + 4x^3 \right) dx + \int -3y^2\,dy = c$

i.e., $y^2 \dfrac{e^{xy^2}}{y^2} + x^4 - y^3 = c$

Thus $e^{xy^2} + x^4 - y^3 = c$, is the required solution

3. [illegible]

$$\left(1 + e^{x/y}\right) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$

>> Let $M = 1 + e^{x/y}$ and $N = e^{x/y}\left(1 - \dfrac{x}{y}\right)$

$\therefore \dfrac{\partial M}{\partial y} = e^{x/y}\left(-\dfrac{x}{y^2}\right)$; $\dfrac{\partial N}{\partial x} = e^{x/y}\cdot\left(-\dfrac{1}{y}\right) + \left(1 - \dfrac{x}{y}\right)e^{x/y}\cdot\dfrac{1}{y}$

i.e., $\dfrac{\partial M}{\partial y} = -\dfrac{xe^{x/y}}{y^2}$; $\dfrac{\partial N}{\partial x} = -\dfrac{1}{y}e^{x/y} + \dfrac{1}{y}e^{x/y} - \dfrac{x}{y^2}e^{x/y} = -\dfrac{xe^{x/y}}{y^2}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int \left(1 + e^{x/y}\right) dx + \int 0\,dy = c$

i.e., $x + \dfrac{e^{x/y}}{(1/y)} = c$

Thus $\boldsymbol{x + y\,e^{x/y} = c}$, is the required solution

Remark *Refer to Problem 15, where this Problem is worked by putting* $x = vy$

36. *Solve :* $\left(2+2x^2\sqrt{y}\right)y\,dx + \left(x^2\sqrt{y}+2\right)x\,dy = 0$

>> Let $M = 2y + 2x^2 y^{3/2}$ and $N = x^3\sqrt{y} + 2x$

$$\frac{\partial M}{\partial y} = 2 + 2x^2 \cdot \frac{3}{2}y^{1/2} \text{ and } \frac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2$$

ie., $$\frac{\partial M}{\partial y} = 2 + 3x^2\sqrt{y} \text{ and } \frac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $$\int\left(2y + 2x^2y^{3/2}\right)dx + \int 0\,dy = c$$

Thus $2xy + \dfrac{2}{3}x^3y^{3/2} = c$, is the required solution

37. *Solve :* $\left(x - \dfrac{y}{x^2+y^2}\right)dx + \left(y + \dfrac{x}{x^2+y^2}\right)dy = 0$ *given that* $y = 1$ *when* x

>> Let $M = x - \dfrac{y}{x^2+y^2}$ and $N = y + \dfrac{x}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1) + y\cdot 2y}{(x^2+y^2)^2}, \quad \frac{\partial N}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

ie., $$\frac{\partial M}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2} \text{ and } \frac{\partial N}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie, $$\int\left(x - \frac{y}{x^2+y^2}\right)dx + \int y\,dy = c$$

ie., $$\frac{x^2}{2} - y\cdot\frac{1}{y}\tan^{-1}(x/y) + \frac{y^2}{2} = c$$

ie., $$\frac{x^2}{2} + \frac{y^2}{2} - \tan^{-1}(x/y) = c, \text{ is the general solution}$$

Given that $y = 1$ when $x = 1$, the general solution becomes

$$\frac{1}{2}+\frac{1}{2}-\tan^{-1}(1)=c \quad \text{or} \quad c=1-(\pi/4)$$

Thus $\dfrac{x^2}{2}+\dfrac{y^2}{2}-\tan^{-1}(x/y)=1-(\pi/4)$, is the particular solution

6.35 | Equations reducible to the exact form

Sometimes the given differential equation which is not an exact equation can be transformed into an exact equation by multiplying with some function (*factor*) known as the integrating factor (I.F)

The procedure to find such a factor is as follows

Integrating Factor : Type - 1

Suppose that, for the equation $M\,dx+N\,dy=0$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}, \quad \text{then we take their difference.}$$

The difference $\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}$ should be close to the expression of M or N

If it is so, then we compute $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)$ or $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)$

If $$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x) \quad \text{or} \quad \frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-g(y)$$

then $e^{\int f(x)\,dx}$ or $e^{\int g(y)\,dy}$ is an integrating factor

As already stated the multiplication of this factor into the equation $M\,dx+N\,dy=0$ will make the equation an exact one and we solve this modified equation being an exact equation.

The following basic results will be useful

(i) $e^{\log x}=x$ (ii) $e^{n\log x}=x^n$

Remark *If the given equation is not exact we continue the working and try to find out whether it can be reduced to an exact equation as per the described procedure. Even in the case of non-exact equations it is worth while trying to see whether the equation can be reduced to an exact form.*

WORKED PROBLEMS

38. *Solve :* $(4xy + 3y^2 - x)\,dx + x(x + 2y)\,dy = 0$

[*The first two methods can be ruled out at once for solving this problem*]

>> Let $M = 4xy + 3y^2 - x$ and $N = x(x + 2y) = x^2 + 2xy$

$$\frac{\partial M}{\partial y} = 4x + 6y \text{ and } \frac{\partial N}{\partial x} = 2x + 2y \quad (\textit{The equation is not exact})$$

Consider $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 2x + 4y = 2(x + 2y)$ ... close to N

Now $\quad \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{2(x + 2y)}{x(x + 2y)} = \dfrac{2}{x} = f(x)$

Hence $e^{\int f(x)\,dx}$ is an integrating factor.

ie., $\quad e^{\int f(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\log x} = e^{\log(x^2)} = x^2$

Multiplying the given equation by x^2 we now have,

$$M = 4x^3y + 3x^2y^2 - x^3 \text{ and } N = x^4 + 2x^3y$$

$$\frac{\partial M}{\partial y} = 4x^3 + 6x^2y \text{ and } \frac{\partial N}{\partial x} = 4x^3 + 6x^2y$$

Remark . *Checking the exactness condition as such is not required as the given equat sure to reduce to an exact equation on multiplication with the proper I.F. But it is just a check as it will give an opportunity for one to rectify the mistake in the event of the exa condition not being satisfied.*

Solution of the exact equation is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int\left(4x^3y + 3x^2y^2 - x^3\right)dx + \int 0\,dy = c$

Thus $\quad x^4y + x^3y^2 - \dfrac{x^4}{4} = c$, is the required solution

39. *Solve :* $y(2x - y + 1)\,dx + x(3x - 4y + 3)\,dy = 0$

>> Let $M = y(2x - y + 1)$ and $N = x(3x - 4y + 3)$

ie., $\quad M = 2xy - y^2 + y$ and $N = 3x^2 - 4xy + 3x$

$$\frac{\partial M}{\partial y} = 2x - 2y + 1\,, \quad \frac{\partial N}{\partial x} = 6x - 4y + 3$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4x + 2y - 2 = -2(2x - y + 1)\ldots \text{near to } M$$

Now, $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{-2(2x-y+1)}{y(2x-y+1)}=-\dfrac{2}{y}=g(y)$

Hence $I \cdot F = e^{-\int g(y)\,dy} = e^{\int \frac{2}{y} dy} = e^{2\log y} = e^{\log(y^2)} = y^2$

Multiplying the given equation with y^2 we now have,

$$M = 2xy^3 - y^4 + y^3 \quad \text{and} \quad N = 3x^2y^2 - 4xy^3 + 3xy^2$$

$$\frac{\partial M}{\partial y} = 6xy^2 - 4y^3 + 3y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy^2 - 4y^3 + 3y^2$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int\left(2xy^3 - y^4 + y^3\right)dx + \int 0\,dy = c$

Thus $x^2y^3 - xy^4 + xy^3 = c$, is the required solution.

40. *Solve* $(x^2+y^2+x)\,dx + xy\,dy = 0$

>> Let $M = x^2+y^2+x$ and $N = xy$

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y \ldots \text{near to } N.$$

Now $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{y}{xy}=\dfrac{1}{x}=f(x)$

Hence $I \cdot F = e^{\int f(x)\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying the given equation by x, we now have,

$$M = x^3 + xy^2 + x^2 \quad \text{and} \quad N = x^2y$$

$$\frac{\partial M}{\partial y} = 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 2xy$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

i.e., $\int\left(x^3 + xy^2 + x^2\right)dx + \int 0\,dy = c$

Thus $\dfrac{x^4}{4} + \dfrac{x^2y^2}{2} + \dfrac{x^3}{3} = c$, is the required solution.

41. *Solve :* $y\,(2xy+1)\,dx - x\,dy = 0$

>> Let $M = y\,(2xy+1)$ and $N = -x$

$$\frac{\partial M}{\partial y} = 4xy+1 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy+2 = 2\,(2xy+1) \ldots \text{near to } M.$$

Now, $$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2\,(2xy+1)}{y\,(2xy+1)} = \frac{2}{y} = g\,(y)$$

Hence $I \cdot F = e^{-\int g\,(y)\,dy} = e^{-\int \frac{2}{y}\,dy} = e^{-2\log y} = y^{-2} = \dfrac{1}{y^2}$

Multiplying the given equation with $1/y^2$, we now have

$$M = 2x + \frac{1}{y} \quad \text{and} \quad N = -\frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

The solution is $\int M\,dx + \int N\,(y)\,dy = c$

ie, $$\int\left(2x + \frac{1}{y}\right)dx + \int 0\,dy = c$$

Thus $x^2 + \dfrac{x}{y} = c$, is the requred solution.

42. *Solve :* $y\,(x+y)\,dx + (x+2y-1)\,dy = 0$

>> Let $M = xy + y^2$ and $N = x + 2y - 1$

$$\frac{\partial M}{\partial y} = x + 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y - 1 \ldots \text{near to } N$$

Now $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{x+2y-1}{x+2y-1} = 1 = f(x)$$

[Though the [illegible] that 1 can be regarded [illegible] function [illegible] f(x) [illegible] the difference in the partial derivatives is divided by N *]*

Hence $I \cdot F = e^{\int f\,(x)\,dx} = e^{\int 1\,dx} = e^x$

Multiplying the given equation by e^x, we now have

$$M = e^x (xy + y^2) \quad \text{and} \quad N = e^x (x + 2y - 1)$$

$$\frac{\partial M}{\partial y} = e^x x + e^x \cdot 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = e^x (x + 2y - 1) + e^x$$

ie., $$\frac{\partial M}{\partial y} = x e^x + 2 e^x y \quad \text{and} \quad \frac{\partial N}{\partial x} = x e^x + 2 e^x y$$

The solution is $\int M\, dx + \int N(y)\, dy = c$

ie., $$y \int x e^x\, dx + y^2 \int e^x\, dx + \int 0\, dy = c$$

ie., $$y (x e^x - e^x) + y^2 e^x = c$$

Thus $e^x (xy + y^2 - y) = c$, is the required solution

43. *Solve :* $(8xy - 9y^2)\, dx + 2(x^2 - 3xy)\, dy = 0$

[*It may be observed that the given equation is a homogeneous one and before one ventures to use the substitution $y = vx$, the option of exact equation is worth while trying.*]

>> Let $M = 8xy - 9y^2$ and $N = 2x^2 - 6xy$

$$\frac{\partial M}{\partial y} = 8x - 18y \quad \text{and} \quad \frac{\partial N}{\partial x} = 4x - 6y$$

$$\therefore \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x - 12y = 4(x - 3y) \text{ is near to } N = 2x(x - 3y)$$

Now $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4(x - 3y)}{2x(x - 3y)} = \frac{2}{x} = f(x)$$

Hence $$I \cdot F = e^{\int f(x)\, dx} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = x^2$$

Multiplying the given equation by x^2 we now have

$$M = 8x^3 y - 9x^2 y^2 \quad \text{and} \quad N = 2x^4 - 6x^3 y$$

$$\frac{\partial M}{\partial y} = 8x^3 - 18x^2 y \quad \text{and} \quad \frac{\partial N}{\partial x} = 8x^3 - 18x^2 y$$

The solution is $\int M\, dx + \int N(y)\, dy = c$

ie., $$\int \left(8x^3 y - 9x^2 y^2\right) dx + \int 0\, dy = c$$

ie., $2x^4 y - 3x^3 y^2 = c$, is the required solution.

Integrating Factor : Type - 2

If the given equation $M\,dx + N\,dy = 0$ is of the form

$$y f(xy)\,dx + x g(xy)\,dy = 0$$

then $\dfrac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$

Remark *It should be clearly noted that 'f' and 'g' in function f (xy) r*
variables x, y

WORKED PROBLEMS

44. Solve : $y(xy+1)\,dx + x(1 - xy)\,dy = 0$

>> The given equation is of the form :

$$y f(xy)\,dx + x g(xy)\,dy = 0 \text{ where}$$

$$M = y f(xy) = xy^2 + y \text{ and } N = x g(xy) = x - x^2 y$$

Now $\quad Mx - Ny = x^2y^2 + xy - xy + x^2y^2 = 2x^2y^2$

$$\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2} \text{ is the integrating factor.}$$

Multiplying the given equation with $1/2x^2y^2$ it becomes an exact equation w
now have,

$$M = \frac{1}{2x^2y^2}(xy^2 + y) \text{ and } N = \frac{1}{2x^2y^2}(x - x^2y)$$

ie., $\quad M = \dfrac{1}{2x} + \dfrac{1}{2x^2y} \quad$ and $\quad N = \dfrac{1}{2xy^2} - \dfrac{1}{2y}$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \displaystyle\int \left(\frac{1}{2x} + \frac{1}{2x^2y}\right) dx + \int -\frac{1}{2y}\,dy = c$

ie., $\quad \dfrac{1}{2}\log x - \dfrac{1}{2xy} - \dfrac{1}{2}\log y = c$

or $\quad \log(x/y) - \dfrac{1}{xy} = 2c$, is the required solution

45. *Solve :* $y\,(1 + xy + x^2y^2)\,dx + x\,(1 - xy + x^2y^2)\,dy = 0$

>> The equation is of the form $y f(xy)\,dx + x g(xy)\,dy = 0$ where,

$$M = y f(xy) = y + xy^2 + x^2y^3 \text{ and}$$

$$N = x g(xy) = x - x^2y + x^3y^2$$

Now $Mx - Ny = (xy + x^2y^2 + x^3y^3) - (xy - x^2y^2 + x^3y^3) = 2x^2y^2$

$$\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2} \text{ is the I.F}$$

Multiplying the given equation with $1/2x^2y^2$ it becomes an exact equation where we now have,

$$M = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \quad \text{and} \quad N = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

The solution is given by $\int M\,dx + \int N(y)\,dy = c$

ie., $$\int\left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}\right)dx + \int -\frac{1}{2y}\,dy = c$$

ie., $$-\frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

or $$\log(x/y) + xy - \frac{1}{xy} = 2c \text{ , is the required solution}$$

46. *Solve :* $y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0$

>> The equation of the form $y f(xy)\,dx + x g(xy)\,dy = 0$ where,

$$M = xy^2 + 2x^2y^3 \quad \text{and} \quad N = x^2y - x^3y^2$$

Now $Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3$

Thus $1/3x^3y^3$ is the I.F. Multiplying the given equation by this I.F we have an exact equation where we now have,

$$M = \frac{1}{3x^2y} + \frac{2}{3x} \quad \text{and} \quad N = \frac{1}{3xy^2} - \frac{1}{3y}$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$$

ie. $$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

or $$-\frac{1}{xy}+\log x^2-\log y = 3c$$

Thus $\log(x^2/y) - 1/xy = 3c$, is the required solution

47. *Solve the following equation*

$$\left[\, xy\sin(xy)+\cos(xy)\,\right]y\,dx+\left[\, xy\sin(xy)-\cos(xy)\,\right]x\,dy = 0$$

>> The equation is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$

$$M = xy^2\sin(xy)+y\cos(xy) \quad\text{and}\quad N = x^2y\sin(xy) - x\cos(xy)$$

Now $Mx - Ny = 2xy\cos(xy)$

Thus $$\frac{1}{2xy\cos(xy)} = \frac{\sec(xy)}{2xy} \text{ is the I.F.}$$

Multiplying the given equation with this I.F we have,

$$M = \frac{y}{2}\tan(xy)+\frac{1}{2x} \quad\text{and}\quad N = \frac{x}{2}\tan(xy)-\frac{1}{2y}$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $$\int\left[\frac{y}{2}\tan(xy)+\frac{1}{2x}\right]dx+\int -\frac{1}{2y}\,dy = c$$

ie., $$\frac{y}{2}\,\frac{\log[\sec(xy)]}{y}+\frac{1}{2}\log x-\frac{1}{2}\log y = c$$

or $$\log\sec(xy)+\log(x/y) = 2c$$

or $$\log\left[\sec(xy)\cdot(x/y)\right] = \log k \quad (say)$$

$\Rightarrow$ $$\sec(xy)\cdot(x/y) = k$$

Thus $x\sec(xy) = ky$, is the required solution

Integrating factor : Type - 3

If the given equation $M\,dx + N\,dy = 0$ is of the form

$$x^{k_1}y^{k_2}(c_1\,y\,dx + c_2\,x\,dy) + x^{k_3}y^{k_4}(c_3\,y\,dx + c_4\,x\,dy) = 0$$

where k_i and c_i $(i = 1$ to $4)$ are constants then $x^a y^b$ is an integrating factor constants a and b are determined such that the condition for an exact equatio satisfied.

Remark *It may be observed that in this type, the terms M and N of the equations a the form $x^a y^b$ only where a and b are constants. In such a case we multiply the equ*

simply by $x^a\ y^b$ and find $a,\ b$ such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Later we solve the equation by using the values of a and b so obtained.

WORKED PROBLEMS

48. *Solve :* $x\,(\,3y\,dx + 2x\,dy\,) + 8y^4\,(\,y\,dx + 3x\,dy\,) = 0$

>> The given equation can be rearranged as

$$(\,3xy + 8y^5\,)\,dx + (\,2x^2 + 24xy^4\,)\,dy = 0 \qquad \ldots (1)$$

[This equation is not an exact equation and the first two types of finding the I.F. can be ruled out easily. Observe that the equation contains only terms of the form $x^a\,y^b$]

Multiplying (1) by $x^a\,y^b$ we have

$$M = 3\,x^{a+1}\,y^{b+1} + 8\,x^a\,y^{b+5}$$

$$N = 2\,x^{a+2}\,y^b + 24\,x^{a+1}\,y^{b+4}$$

We shall find a and b such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$3x^{a+1}\,(\,b+1\,)\,y^b + 8x^a\,(\,b+5\,)\,y^{b+4} = 2\,(\,a+2\,)\,x^{a+1}\,y^b + 24\,(\,a+1\,)\,x^a\,y^{b+4}$$

$\Rightarrow \quad 3\,(\,b+1\,) = 2\,(\,a+2\,)$ and $8\,(\,b+5\,) = 24\,(\,a+1\,)$

or $\quad 2a - 3b = -1$ and $3a - b = 2$

By solving we get $a = 1,\ b = 1.$

We now have $M = 3x^2\,y^2 + 8xy^6$ and $N = 2x^3\,y + 24x^2\,y^5$

The solution is $\int M\,dx + \int N\,(\,y\,)\,dy = c$

ie., $\quad \int \left(3x^2\,y^2 + 8xy^6\right) dx + \int 0\ dy = c$

Thus $\quad x^3\,y^2 + 4\,x^2\,y^6 = c\,,\quad$ is the required solution.

49. *Solve :* $x\,(\,4y\,dx + 2x\,dy\,) + y^3\,(\,3y\,dx + 5x\,dy\,) = 0$

>> We have $(\,4xy + 3y^4\,)\,dx + (\,2x^2 + 5xy^3\,)\,dy = 0$

Multiplying the equation by $x^a\,y^b$ we have,

$$M = 4x^{a+1}\,y^{b+1} + 3x^a\,y^{b+4} \quad \text{and} \quad N = 2x^{a+2}\,y^b + 5x^{a+1}\,y^{b+3}$$

$$\frac{\partial M}{\partial y} = 4\,(\,b+1\,)\,x^{a+1}\,y^b + 3\,(\,b+4\,)\,x^a\,y^{b+3}$$

$$\frac{\partial N}{\partial x} = 2(a+2)x^{a+1}y^{b} + 5(a+1)x^{a}y^{b+3}$$

We have to find a and b such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\Rightarrow \qquad 4(b+1) = 2(a+2)$ and $3(b+4) = 5(a+1)$

ie., $\qquad a = 2b$ and $5a - 3b = 7$

By solving we get $a = 2$ and $b = 1$

We now have, $M = 4x^3y^2 + 3x^2y^5$ and $N = 2x^4y + 5x^3y^4$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\qquad \int \left(4x^3y^2 + 3x^2y^5\right)dx + \int 0\,dy = c$

Thus $\quad x^4y^2 + x^3y^5 = c$, is the required solution.

50. *Solve* : $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

[*We cannot find I.F. by the earlier described methods*]

>> Multiplying the given equation by $x^a y^b$ we have

$$M = x^a y^{b+2} + 2x^{a+2}y^{b+1} \quad \text{and} \quad N = 2x^{a+3}y^{b} - x^{a+1}y^{b+1}$$

$$\frac{\partial M}{\partial y} = (b+2)x^{a}y^{b+1} + 2(b+1)x^{a+2}y^{b}$$

$$\frac{\partial N}{\partial x} = 2(a+3)x^{a+2}y^{b} - (a+1)x^{a}y^{b+1}$$

Let us find a and b such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\Rightarrow \qquad (b+2) = -(a+1)$ and $2(b+1) = 2(a+3)$

ie., $\qquad a + b = -3$ and $a - b = -2$

By solving we get $a = -5/2$ and $b = -1/2$.

We now have,

$$M = x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2} \quad \text{and} \quad N = 2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\qquad \int \left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \int 0\,dy = c$

i.e., $\dfrac{x^{-3/2}}{-3/2}\, y^{3/2} + 2\,\dfrac{x^{1/2}}{1/2}\, y^{1/2} = c$

i.e., $\dfrac{-2}{3}\, x^{-3/2}\, y^{3/2} + 4x^{1/2}\, y^{1/2} = c$

or $-x^{-3/2}\, y^{3/2} + 6x^{1/2}\, y^{1/2} = 3c/2$

Thus $6\sqrt{xy} - \sqrt{y^3/x^3} = k$, is the required solution, where $k = 3c/2$

Type - 4 : Exactness by inspection

The preamble for this method is that we should be in a position to recognize certain *standard exact differentials* listed as follows.

E_1. $dx \pm dy = d\,(x \pm y)$

E_2. $x\,dy + y\,dx = d\,(xy)$

E_3. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

E_4. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

E_5. $\dfrac{x\,dy - y\,dx}{x^2 - y^2} = d\left[\dfrac{1}{2}\log\left(\dfrac{x+y}{x-y}\right)\right]$

E_6. $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right] = -d\left[\tan^{-1}\left(\dfrac{x}{y}\right)\right]$

E_7. $\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log(x^2 + y^2)\right] = d\left[\log\sqrt{x^2 + y^2}\right]$

E_8. $\dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = d\left[\sqrt{x^2 + y^2}\right]$

With this preamble we describe the method

The given equation itself can be a combination of various standard exact differentials or we may have to judiciously rearrange the terms of the given equation to match with the exact differentials fully or partly as the case may be. In otherwords the given equation must assume the form

$$c_1\, d\,[f_1(x,\, y)] + c_2\, d\,[f_2(x,\, y)] + \cdots = 0$$

In such a case, the solution can instantly be written as

$c_1 f_1(x,\, y) + c_2 f_2(x,\, y) + \cdots = c$, by integration.

WORKED PROBLEMS

51. *Solve :* $x\,dx + y\,dy - \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$

>> The given equation is equivalent to the form

$$x\,dx + y\,dy - d\left[\tan^{-1}(y/x)\right]$$

Integrating we get,

$$\frac{x^2}{2} + \frac{y^2}{2} - \tan^{-1}(y/x) = c\,, \text{ being the required solution}$$

Remark *Had we not recognized the third term of the given equation as* $d\left[\tan^{-1}\right.$ *then in the normal course we should have written the given equation in the form*

$$\left[x + \frac{y}{x^2+y^2}\right]dx + \left[y - \frac{x}{x^2+y^2}\right]dy = 0$$

and solve by verifying the exactness condition similar to Problem - 37 Furthe
$-d\left[\tan^{-1}(y/x)\right] = d\left[\tan^{-1}(x/y)\right]$ the solution can also be in the form

$$\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1}(x/y) = c$$

52. *Solve :* $\dfrac{y\,dx - x\,dy}{y^2} + (x\,dx + y\,dy) = 0$

>> The given equation is equivalent to the form,

$$d\left(\frac{x}{y}\right) + x\,dx + y\,dy = 0$$

$$\Rightarrow \quad \frac{x}{y} + \frac{x^2}{2} + \frac{y^2}{2} = c, \text{ on integration.}$$

Thus $\quad \dfrac{x}{y} + \dfrac{1}{2}(x^2 + y^2) = c,$ is the required solution

53. *Solve :* $y e^{x/y}\,dx = (x e^{x/y} + y^2)\,dy$

[*It should be carefully noted that though we have* (x/y) *terms in the equation it is* *homogeneous equation*]

>> The given equation can be put in the form,

$$e^{x/y}(y\,dx - x\,dy) = y^2\,dy$$

or $\qquad e^{x/y}\left[\dfrac{y\,dx - x\,dy}{y^2}\right] = dy$

ie., $\qquad e^{x/y}\,d(x/y) = dy$

Integration yields $e^{x/y} = y + c$

Thus $\qquad e^{x/y} - y = c$, is the required solution.

Remark : *The Problem can be solved by verifying the exactness condition and it will be little difficult.*

54. *Solve* : $x\,dx = y(x^2 + y^2 - 1)\,dy$

>> The given equation can be put in the form

$$x\,dx + y\,dy = y(x^2 + y^2)\,dy \quad \text{or} \quad \frac{x\,dx + y\,dy}{x^2 + y^2} = y\,dy$$

ie., $\qquad d\left[\dfrac{1}{2}\log(x^2 + y^2)\right] = y\,dy$

Integrating we get, $\dfrac{1}{2}\log(x^2 + y^2) = \dfrac{y^2}{2} + c$

or $\qquad \log(x^2 + y^2) = y^2 + 2c$ and let $k = 2c$

Thus $\qquad \log(x^2 + y^2) - y^2 = k$, is the required solution

55. *Solve* $[1 + y\tan(xy)]\,dx + [x\tan(xy)]\,dy = 0$

>> The given equation can be put in the form

$$dx + \tan(xy)[y\,dx + x\,dy] = 0$$

ie., $\qquad dx + \tan(xy)\,d(xy) = 0$

Integrating we get $x + \log\sec(xy) = c$, being the required solution

EXERCISES

Solve the following differential equations

1. $3x(xy-2)\,dx+(x^3+2y)\,dy = 0$
2. $(\cos 2y - 3x^2y^2)\,dx+(\cos 2y - 2x\sin 2y - 2x^3y)\,dy = 0$
3. $(6x^2+2xy-2xy\,e^{-x^2})\,dx + (e^{-x^2}+x^2+3y^2)\,dy = 0$
4. $(2x^2+6xy-y^2)\,dx+(3x^2-2xy+y^2)\,dy = 0$
5. $\cos x\,(e^y+1)\,dx+\sin x\,e^y\,dy = 0$ *[solve by two methods]*
6. $\left[4x^3y^2+y\cos(xy)\right]dx+\left[2x^4y+x\cos(xy)\right]dy = 0$
7. $\dfrac{dy}{dx} = \dfrac{y+x-2}{y-x-4}$ *[solve by two methods]*
8. $(xy^2+x-2y+3)\,dx+x^2y\,dy = 2(x+y)\,dy$; $y(1) = 1$
9. $y(x+y+1)\,dx+x(x+3y+2)\,dy = 0$
10. $2(3x^2+2y^3+6y)\,dx+3(x+xy^2)\,dy = 0$ given that $y(1) = 2$
11. $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy = 0$
12. $(x^7y^2+3y)\,dx+(3x^8y-x)\,dy = 0$
13. $(x^2y^2+5xy+2)\,y\,dx+(x^2y^2+4xy+2)\,x\,dy = 0$
14. $y\,dx-x\,dy+3x^2y^2e^{x^3}\,dx = 0$
15. $x\,dy-y\,dx = \cos^2(y/x)\,dx$

ANSWERS

1. $x^3y-3x^2+y^2 = c$
2. $\dfrac{\sin 2y}{2}+x\cos 2y-x^3y^2 = c$
3. $2x^3+x^2y+ye^{-x^2}+y^3 = c$
4. $2x^3+9x^2y-3xy^2+y^3 = k$
5. $\sin x\,(e^y+1) = c$
6. $x^4y^2+\sin(xy) = c$
7. $x^2+2xy-y^2-4x+8y = c$
8. $\dfrac{x^2y^2}{2}+\dfrac{x^2}{2}-2xy+3x-y^2 = 1$
9. $\dfrac{x^2y^2}{2}+xy^3+xy^2 = c$
10. $x^6+x^4y^3+3x^4y = 15$
11. $x^3y^3+x^2 = cy$
12. $2x^7y^3-y^2 = cx^6$

13. $\log(x^5/y^4) = (2/xy) - xy + c$ 14. $(x/y) + e^{x^3} = c$

15. $\tan(y/x) + 1/x = c$

36 Linear Differential Equations

A differential equation is said to be *linear* if the dependent variable and its derivative occurs in the first degree only and they are not multiplied together.

Standard form of a linear equation and its solution

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \dots (1)$$

where P and Q are functions of x only is called a *linear equation in 'y'* and we shall solve the same

Multiplying (1) by $e^{\int P\,dx}$ we have,

$$e^{\int P\,dx}\frac{dy}{dx} + Py\,e^{\int P\,dx} = Q\,e^{\int P\,dx} \quad \text{or} \quad \frac{d}{dx}\left[e^{\int P\,dx}\,y\right] = Q\,e^{\int P\,dx}$$

Integrating w.r.t. x on both sides we have,

$$e^{\int P\,dx}\,y = \int Q\,e^{\int P\,dx}\,dx + c$$

ie., $y\,e^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx + c$ is the solution of linear equation (1)

Linear equation in 'x'

An equation of the form $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y only is called a *linear equation in x.*

Its solution can simply be written by interchanging the role of x and y in the solution obtained already for the linear equation in y.

$$x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$$

is the solution for the linear equation in x.

Working procedure for problems

- The given equation must be first put in the form conformal to the standard form of the linear equation in x or y.
- The expression for P and Q is to be written by simple comparison

- We equip with the I. F $e^{\int P\,dx}$ or $e^{\int P\,dy}$.
- We assume the associated solution and we only need to tackle the R H S the solution to finally arrive at the required solution.

Note *If the given equation is not in the standard form of a linear equation but is in $M(x, y)\,dx + N(x, y)\,dy = 0$ then the linearity of the D E can be recognised following observations.*

1. ***Linear in y*** *M contains 'y' (not y^2, log y etc), N is a function of x only*

2. ***Linear in x*** *N contains 'x' (not x^2, sin x etc), M is a function of y only*

In such a case the equation is transformed into the standard form of the linear equation be solved by assuming the solution

Remark *Many of the problems when the equation is given in the $M(x, y)\,dx + N(x, y)\,dy = 0$ can be solved by the earlier methods. If we are recognise the linearity then we can assume the solution in respect of equations in the standard form of linear equation.*

WORKED PROBLEMS

>> $\frac{dy}{dx} - \frac{2y}{x} = x + x^2$ is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -\frac{2}{x} \text{ and } Q = x + x^2$$

$$e^{\int P\,dx} = e^{-\int 2/x\,dx} = e^{-2\log x} = e^{\log x^{-2}} = 1/x^2$$

The solution is given by $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

i.e., $y \cdot \frac{1}{x^2} = \int (x + x^2)\frac{1}{x^2}\,dx + c$

i.e., $\frac{y}{x^2} = \int \frac{1}{x}\,dx + \int 1\,dx + c$

Thus $\frac{y}{x^2} = \log x + x + c$, is the required solution.

57. *Solve :* $\dfrac{dy}{dx} + y\cot x = \cos x$

>> $\dfrac{dy}{dx} + y\cot x = \cos x$ is of the form :

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \cot x \text{ and } Q = \cos x$$

$$e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x$$

The solution is $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $y\sin x = \int \cos x \cdot \sin x\,dx + c$

ie., $y\sin x = \int \dfrac{\sin 2x}{2}\,dx + c$

Thus $\mathbf{y\sin x = \dfrac{-\cos 2x}{4} + c}$, is the required solution

58. *Solve :* $x\cos x\dfrac{dy}{dx} + (\cos x - x\sin x)\,y = 1$

>> Dividing the given equation through out by $x\cos x$ we have,

$$\frac{dy}{dx} + \left[\frac{\cos x - x\sin x}{x\cos x}\right]y = \frac{1}{x\cos x}$$

This is of the form $\dfrac{dy}{dx} + Py = Q$ where

$$P = \frac{\cos x - x\sin x}{x\cos x} \text{ and } Q = \frac{1}{x\cos x}$$

$$e^{\int P\,dx} = e^{\log(x\cos x)} = x\cos x$$

The solution is $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $y \cdot x\cos x = \int \dfrac{1}{x\cos x} \cdot x\cos x\,dx + c$

Thus $\mathbf{x\,y\cos x = x + c}$, is the required solution.

59. *Solve* $2y' \cos x + 4y \sin x = \sin 2x$ *given that* $y(\pi/3) = 0$

>> Dividing the given equation by $2\cos x$, we have

$$\frac{dy}{dx} + (2\tan x)\,y = \sin x \quad \because \ \sin 2x = 2\cos x \sin x$$

This is of the form $\dfrac{dy}{dx} + Py = Q$ where $P = 2\tan x$ and $Q = \sin x$

$$e^{\int P dx} = e^{\int 2\tan x\,dx} = e^{2\log(\sec x)} = \sec^2 x$$

The solution is $y\,e^{\int P dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad y\sec^2 x = \int \sin x \sec^2 x\,dx + c$

ie., $\quad y\sec^2 x = \int \tan x \sec x\,dx + c$

ie., $\quad y\sec^2 x = \sec x + c$, is the general solution

Consider $y(\pi/3) = 0$. That is $y = 0$ when $x = \pi/3$

Hence (1) becomes $0 = 2 + c$ or $c = -2$

Thus $\quad y\sec^2 x = \sec x - 2$, is the required particular solution

60. *Solve* $\dfrac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$, *if* $y = 0$ *when* $x = \pi/2$

>> $\dfrac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$ is of the form :

$\dfrac{dy}{dx} + Py = Q$, where $P = \cot x$ and $Q = 4x \operatorname{cosec} x$

Now $e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x.$

Solution is given by $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

i.e., $\quad y\sin x = \int 4x \operatorname{cosec} x \cdot \sin x\,dx + c$

i.e., $\quad y\sin x = \int 4x\,dx + c$

i.e., $\quad y\sin x = 2x^2 + c$ is the general solution.

But $\quad y = 0$ when $x = \pi/2$ by data.

$\quad 0 = \pi^2/2 + c \qquad \therefore\ c = -\pi^2/2$

Thus $y\sin x = 2x^2 - (\pi^2/2)$, is the particular solution

61. Solve $\frac{dy}{dx} + 3x^2 y = x^5 e^{x^3}$

>> The given equation is of the form $\frac{dy}{dx} + Py = Q$

where $P = 3x^2$ and $Q = x^5 e^{x^3}$ $\quad \therefore \quad e^{\int P\,dx} = e^{\int 3x^2\,dx} = e^{x^3}$

The solution is $y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

ie., $\quad y\, e^{x^3} = \int x^5 e^{x^3} \cdot e^{x^3}\, dx + c$

ie., $\quad y\, e^{x^3} = \int x^5 e^{2x^3}\, dx + c \qquad \ldots (1)$

Put $\quad x^3 = t \qquad \therefore \; 3x^2\, dx = dt$ or $3x^5\, dx = x^3\, dt$

Hence $x^5\, dx = \frac{t\, dt}{3}$

Thus (1) becomes $y\, e^{x^3} = \frac{1}{3} \int t\, e^{2t}\, dt + c$

ie., $\quad y\, e^{x^3} = \frac{1}{3} \left\{ t\, \frac{e^{2t}}{2} - \int \frac{e^{2t}}{2} \cdot 1\, dt \right\} + c$

ie., $\quad y\, e^{x^3} = \frac{1}{3} \left\{ \frac{t\, e^{2t}}{2} - \frac{e^{2t}}{4} \right\} + c$

ie., $\quad y\, e^{x^3} = \frac{e^{2t}}{12} (2t - 1) + c$

Thus $\quad y\, e^{x^3} = \frac{e^{2x^3}}{12} (2x^3 - 1) + c$, is the required solution

62. Solve : $x(1 - x^2)\frac{dy}{dx} + (2x^2 - 1)\, y = x^3$

>> Dividing the given equation throughout by $x(1 - x^2)$ we have,

$$\frac{dy}{dx} + \left\{ \frac{2x^2 - 1}{x(1 - x^2)} \right\} y = \frac{x^2}{(1 - x^2)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{2x^2 - 1}{x(1 - x^2)} \quad \text{and} \quad Q = \frac{x^2}{1 - x^2}$$

$\int P\,dx$ has to be found first, by resolving P into partial fractions

Let $$\frac{2x^2-1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$\Rightarrow$ $$2x^2-1 = A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$$

Put $x = 0$: $-1 = A(1)$ $\therefore$ $A = -1$

Put $x = 1$: $1 = B(2)$ $\therefore$ $B = 1/2$

Put $x = -1$: $1 = C(-2)$ $\therefore$ $C = -1/2$

$$\int P\,dx = -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{1-x}\,dx - \frac{1}{2}\int \frac{1}{1+x}\,dx$$

$$= -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)$$

$$= -\left[\log x + \log\sqrt{1-x} + \log\sqrt{1+x}\right]$$

$$= -\log(x\sqrt{1-x^2})$$

Hence $$e^{\int P\,dx} = e^{-\log[x\sqrt{1-x^2}]} = \frac{1}{x\sqrt{1-x^2}}$$

The solution is $$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$$

ie., $$\frac{y}{x\sqrt{1-x^2}} = \int \frac{x^2}{(1-x^2)} \cdot \frac{1}{x\sqrt{1-x^2}}\,dx + c$$

ie., $$\frac{y}{x\sqrt{1-x^2}} = \int \frac{x}{(1-x^2)^{3/2}}\,dx + c$$

Put $(1-x^2) = t$ $\therefore$ $-2x\,dx = dt$ or $x\,dx = dt/-2$

$$\frac{y}{x\sqrt{1-x^2}} = \frac{-1}{2}\int \frac{dt}{t^{3/2}} + c$$

ie., $$\frac{y}{x\sqrt{1-x^2}} = \frac{-1}{2}\,\frac{t^{-1/2}}{(-1/2)} + c$$

ie., $$\frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + c$$

or $y = x + cx\sqrt{1-x^2}$, is the required solution

63. *Solve* $\sqrt{1-y^2}\,dx = (\sin^{-1}y - x)\,dy$

In the given equation, observe that M is a function of y and N contains x (x dy is present in the equation). Hence the equation is linear in x.

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin^{-1}y - x}{\sqrt{1-y^2}}$$

or $\quad \dfrac{dx}{dy} + \dfrac{x}{\sqrt{1-y^2}} = \dfrac{\sin^{-1}y}{\sqrt{1-y^2}}$ and this is of the form

$$\frac{dx}{dy} + Px = Q, \quad \text{where} \quad P = \frac{1}{\sqrt{1-y^2}} \quad \text{and} \quad Q = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$$

$$e^{\int P\,dy} = e^{\sin^{-1}y}$$

The solution is $\quad x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

i.e., $\quad x\,e^{\sin^{-1}y} = \int \dfrac{\sin^{-1}y}{\sqrt{1-y^2}}\,e^{\sin^{-1}y}\,dy + c$

Put $\quad \sin^{-1}y = t \Rightarrow \dfrac{1}{\sqrt{1-y^2}}\,dy = dt$

$$x\,e^{\sin^{-1}y} = \int t\,e^t\,dt + c$$

i.e. $\quad x\,e^{\sin^{-1}y} = t\,e^t - e^t + c$, on integration by parts

Thus $\quad x\,e^{\sin^{-1}y} = e^{\sin^{-1}y}\left(\sin^{-1}y - 1\right) + c$, is the required solution

64. *Solve* : $(1+y^2)\,dx + (x - \tan^{-1}y)\,dy = 0$

This problem is very much similar to the previous one. Observing the same features in this equation we recognize the equation as a linear equation in x.

We have $\quad \dfrac{dx}{dy} = \dfrac{\tan^{-1}y - x}{1+y^2}$

or $\quad \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1}y}{1+y^2}$

This is of the form $\dfrac{dx}{dy} + Px = Q$

Here $\quad P = \dfrac{1}{1+y^2}$ and $Q = \dfrac{\tan^{-1}y}{1+y^2}$ and $e^{\int P\,dy} = e^{\tan^{-1}y}$

The solution is given by $\quad x\, e^{\int P dy} = \int Q\, e^{\int P dy}\, dy + c$

ie., $\quad x\, e^{\tan^{-1} y} = \int \dfrac{\tan^{-1} y}{1+y^2}\, e^{\tan^{-1} y}\, dy + c$

By putting $\tan^{-1} y = t$, we obtain (*As in the previous problem*)

$$x\, e^{\tan^{-1} y} = e^{\tan^{-1} y}\,(\tan^{-1} y - 1) + c, \text{ which is the required solution}$$

65. *Solve :* $(1 + xy)\,dx - (1 + x^2)\,dy = 0,\ x = 1,\ y = 0$

(Observe that N is a function of x only and M contains y. Thus the equation is linear in

>> We have $\ (1 + xy)\,dx = (1 + x^2)\,dy \ $ or $\ \dfrac{dy}{dx} = \dfrac{1 + xy}{1 + x^2}$

ie., $\quad \dfrac{dy}{dx} - \dfrac{xy}{1+x^2} = \dfrac{1}{1+x^2}$

This equation is of the form $\dfrac{dy}{dx} + Py = Q$, where

$$P = \frac{-x}{1+x^2} \quad \text{and} \quad Q = \frac{1}{1+x^2}$$

$$e^{\int P\, dx} = e^{-\int \frac{x\, dx}{1+x^2}} = e^{-\frac{1}{2}\log(1+x^2)} = \frac{1}{\sqrt{1+x^2}}$$

The solution is $\quad y\, e^{\int P\, dx} = \int Q\, e^{\int P\, dx}\, dx + c$

ie., $\quad \dfrac{y}{\sqrt{1+x^2}} = \int \dfrac{1}{(1+x^2)} \cdot \dfrac{1}{\sqrt{1+x^2}}\, dx + c$

ie., $\quad \dfrac{y}{\sqrt{1+x^2}} = \int \dfrac{dx}{(1+x^2)^{3/2}} + c$

Put $\quad x = \tan\theta \qquad \therefore \quad dx = \sec^2\theta\, d\theta$

Also $\quad (1+x^2)^{3/2} = (1+\tan^2\theta)^{3/2} = \sec^3\theta$

$$\frac{y}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta + c \quad \text{or} \quad \frac{y}{\sqrt{1+x^2}} = \int \cos\theta\, d\theta + c$$

ie., $\quad \dfrac{y}{\sqrt{1+x^2}} = \sin\theta + c$

But $\tan\theta = x \Rightarrow \cot\theta = 1/x$ and $1 + \cot^2\theta = 1 + (1/x^2)$

ie., $\quad \operatorname{cosec}^2 \theta = \dfrac{x^2+1}{x^2}$ or $\operatorname{cosec} \theta = \dfrac{\sqrt{x^2+1}}{x}$

$$\sin \theta = \frac{x}{\sqrt{x^2+1}} \quad \ldots (2)$$

Using (2) in (1) we get the general solution,

$$\frac{y}{\sqrt{1+x^2}} = \frac{x}{\sqrt{x^2+1}} + c \quad \ldots (3)$$

But $x = 1$ and $y = 0$ and hence (3) will give us $c = -1/\sqrt{2}$

Hence (3) becomes,

$$\frac{y}{\sqrt{1+x^2}} = \frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}$$

Thus $\quad y = x - \sqrt{1+x^2/2}$, is the required particular solution

66. Sol

>> We have $dx = (e^{-y} \sec^2 y - x)\, dy$ or $\dfrac{dx}{dy} + x = e^{-y} \sec^2 y$

This is of the form $\dfrac{dx}{dy} + Px = Q$, where

$$P = 1 \text{ and } Q = e^{-y} \sec^2 y \quad \therefore \quad e^{\int P\,dy} = e^y$$

The solution $x\, e^{\int P\,dy} = \int Q\, e^{\int P\,dy}\, dy + c$ becomes

$$x\, e^y = \int e^{-y} \sec^2 y\, e^y\, dy + c$$

Thus $\quad x\, e^y = \tan y + c$, is the required solution.

Observe that M is a function of y and N contains x ($x\,dy$ is present). The equation is linear in x.

>> We have $\quad y\,dx = (xy + 2 - 3x)\, dy$

ie., $\quad \dfrac{dx}{dy} = \dfrac{xy+2-3x}{y}$ or $\dfrac{dx}{dy} = \dfrac{x(y-3)+2}{y}$

ie., $\quad \dfrac{dx}{dy} - \dfrac{x(y-3)}{y} = \dfrac{2}{y}$

This equation is of the form $\dfrac{dx}{dy} + Px = Q$, where

$$P = -\frac{(y-3)}{y} = \left(\frac{3}{y}-1\right) \text{ and } Q = \frac{2}{y}$$

Hence $\int P\,dy = \int\left(\frac{3}{y}-1\right)dy = 3\log y - y$

$$\therefore \quad e^{\int P\,dy} = e^{(3\log y - y)} = e^{3\log y}\cdot e^{-y} = y^3 e^{-y}$$

The solution is $x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $\quad x\,y^3 e^{-y} = \int \frac{2}{y}\cdot y^3 e^{-y}\,dy + c$

ie., $\quad x\,y^3 e^{-y} = 2\int y^2 e^{-y}\,dy + c$

Applying Bernoulli's generalized rule of integration by parts we get,

$$x\,y^3 e^{-y} = 2\left\{y^2(-e^{-y}) - (2y)(e^{-y}) + 2(-e^{-y})\right\} + c$$

ie., $\quad x\,y^3 e^{-y} = -2e^{-y}(y^2+2y+2) + c$

Thus $\quad x\,y^3 = -2(y^2+2y+2) + c\,e^{y}$, is the required solution

68. Solve $(x + \tan y)\,dy = \sin 2y\,dx$

>> We have $\dfrac{dx}{dy} = \dfrac{x+\tan y}{\sin 2y}$

ie., $\quad \dfrac{dx}{dy} - x\,\operatorname{cosec} 2y = \dfrac{\tan y}{\sin 2y}$

This equation is of the form $\dfrac{dx}{dy} + Px = Q$, where

$$P = -\operatorname{cosec} 2y \text{ and } Q = \frac{\tan y}{\sin 2y} = \frac{\sin y}{\cos y \cdot 2\sin y\cos y} = \frac{1}{2}\sec^2 y$$

$$\therefore \quad e^{\int P\,dy} = e^{\int -\operatorname{cosec} 2y\,dy} = e^{-\frac{1}{2}\log(\tan y)} = \frac{1}{\sqrt{\tan y}}$$

where we have used $\int \operatorname{cosec} x\,dx = \log\tan(x/2)$

The solution is $x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $\quad \dfrac{x}{\sqrt{\tan y}} = \int \dfrac{1}{2}\sec^2 y.\ \dfrac{1}{\sqrt{\tan y}}\,dy + c$

Put $\quad \tan y = t \quad \therefore\ \sec^2 y\,dy = dt$

Hence $\dfrac{x}{\sqrt{\tan y}} = \dfrac{1}{2} \displaystyle\int \dfrac{dt}{\sqrt{t}} + c$

ie., $\dfrac{x}{\sqrt{\tan y}} = \dfrac{1}{2} \dfrac{t^{1/2}}{(1/2)} + c$

ie., $\dfrac{x}{\sqrt{\tan y}} = \sqrt{\tan y} + c$

or $\boldsymbol{x = \tan y + c\sqrt{\tan y}}$, **is the required solution.**

69.

Observe that N is a function of x and M contains y. The equation is linear in y

>> We have $dy - dx = \dfrac{2xy}{1+x^2}\,dx$

ie., $dy = \left(\dfrac{2xy}{1+x^2} + 1\right) dx$

ie., $\dfrac{dy}{dx} - \dfrac{2xy}{1+x^2} = 1$

This equation is of the form $\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{-2x}{1+x^2}$ and $Q = 1$

Hence $\int P\,dx = \displaystyle\int \dfrac{-2x}{1+x^2}\,dx = -\log(1+x^2)$

$$e^{\int P\,dx} = e^{-\log(1+x^2)} = \frac{1}{1+x^2}$$

The solution is $y\,e^{\int P\,dx} = \displaystyle\int Q\,e^{\int P\,dx}\,dx + c$

ie., $\dfrac{y}{1+x^2} = \displaystyle\int 1 \cdot \dfrac{1}{1+x^2}\,dx + c$

Thus $\boldsymbol{\dfrac{y}{1+x^2} = \tan^{-1} x + c}$, **is the required solution.**

70.

>> We have $\dfrac{dy}{dx} = \dfrac{2\log x - y}{x\log x}$ or $\dfrac{dy}{dx} + \dfrac{y}{x\log x} = \dfrac{2}{x}$ is linear in y of the form

$\dfrac{dy}{dx} + Py = Q$, **where** $P = \dfrac{1}{x\log x}$ and $Q = \dfrac{2}{x}$

Hence $\int P\,dx = \displaystyle\int \dfrac{1/x}{\log x}\,dx = \log(\log x)$

$\therefore \qquad e^{\int P\,dx} = e^{\log(\log x)} = \log x$

The solution is $\quad y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\qquad y \log x = \int \frac{2}{x} \cdot \log x\,dx + c$

Put $\qquad \log x = t \quad \therefore \quad \frac{1}{x}\,dx = dt$

Hence $\quad y \log x = \int 2t\,dt + c \quad$ or $\quad y \log x = t^2 + c$

Thus $\qquad y \log x = (\log x)^2 + c,$ is the required solution

>> We have $\frac{dr}{d\theta} + (2\cot\theta)\,r = -\sin 2\theta$

Here $\qquad P = P(\theta) = 2\cot\theta,\ Q = Q(\theta) = -\sin 2\theta = -2\sin\theta\cos\theta$

$\qquad \int P\,d\theta = \int 2\cot\theta\,d\theta = 2\log(\sin\theta) = \log(\sin^2\theta)$

$\therefore \qquad e^{\int P\,d\theta} = e^{\log(\sin^2\theta)} = \sin^2\theta$

The solution is $\quad r\,e^{\int P\,d\theta} = \int Q\,e^{\int P d\theta}\,d\theta + c$

ie., $\qquad r\sin^2\theta = \int(-2\sin\theta\cos\theta)\sin^2\theta\,d\theta + c$

ie., $\qquad r\sin^2\theta = -2\int \sin^3\theta\cos\theta\,d\theta + c$

Put $\qquad \sin\theta = t \quad \therefore \quad \cos\theta\,d\theta = dt$

Hence $\quad r\sin^2\theta = -2\int t^3\,dt + c$

ie., $\qquad r\sin^2\theta = -\frac{t^4}{2} + c \qquad$ or $\qquad r\sin^2\theta = \frac{-\sin^4\theta}{2} + c$

Thus $\qquad 2r\sin^2\theta + \sin^4\theta = 2c,$ is the required solution

6.37 Equations reducible to the linear form

Form (i) $f'(y)\dfrac{dy}{dx} + P f(y) = Q$, where P and Q are functions of x

We put $f(y) = t \quad \therefore \quad f'(y)\dfrac{dy}{dx} = \dfrac{dt}{dx}$

The given equation becomes $\dfrac{dt}{dx} + P t = Q$ which is a linear equation in t

Similarly $f'(x)\dfrac{dx}{dy} + P f(x) = Q$, where P and Q are functions of y can be reduced to the linear form by putting $f(x) = t$.

Form (ii). $\dfrac{dy}{dx} + P y = Q y^n$, where P and Q are functions of x

This equation is called as **Bernoulli's equation** in y

We first divide the equation throughout by y^n to obtain

$$\frac{1}{y^n}\frac{dy}{dx} + P y^{1-n} = Q \qquad \ldots (1)$$

Put $\quad y^{1-n} = t \quad \therefore \quad (1-n)\, y^{-n}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or $$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dt}{dx}$$

Hence (1) becomes, $\dfrac{1}{(1-n)}\dfrac{dt}{dx} + P t = Q$

or $\dfrac{dt}{dx} + (1-n)P\, t = (1-n)\,Q$ which is a linear equation in t

Similarly $\dfrac{dx}{dy} + P x = Q x^n$, where P and Q are function of y is called **Bernoulli's equation** in x. We first divide by x^n and later put $x^{1-n} = t$ to obtain a linear equation in t.

WORKED PROBLEMS

>> We have $e^y\dfrac{dy}{dx} + e^y = e^x \qquad \ldots (1)$

Put $\quad e^y = t \quad \therefore \quad e^y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Hence (1) becomes $\dfrac{dt}{dx} + t = e^x$

This equation is linear in t of the form $\dfrac{dt}{dx} + Pt = Q$, where

$P = 1$ and $Q = e^x$. $\quad \therefore \quad e^{\int P\,dx} = e^x$

The solution is $\; t\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

ie., $\quad t\, e^x = \int e^x \cdot e^x\, dx + c$

ie., $\quad e^y\, e^x = \int e^{2x}\, dx + c$

Thus $\quad e^{x+y} = \dfrac{e^{2x}}{2} + c,\;$ **is the required solution.**

Observing $\dfrac{dy}{dx}$ *in the equation, we shall get rid off* $\cos y$ *in the R H S of the given equ*

>> Dividing the given equation by $\cos y$ we have,

$$\sec y\, \tan y\, \frac{dy}{dx} + \sec y\, \tan x = \cos^2 x$$

Now put $\sec y = t \quad \therefore \quad \sec y\, \tan y\, \dfrac{dy}{dx} = \dfrac{dt}{dx}$

Hence (1) becomes $\dfrac{dt}{dx} + t \tan x = \cos^2 x$

This equation is of the form $\dfrac{dt}{dx} + Pt = Q$, where we have,

$$P = \tan x \text{ and } Q = \cos^2 x.$$

$$e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x$$

The solution is $\; t\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

ie., $\quad \sec y\, \sec x = \int \cos^2 x \cdot \sec x\, dx + c$

ie., $\quad \sec y \cdot \sec x = \int \cos x\, dx + c$

Thus $\quad \sec y\, \sec x = \sin x + c,\;$ is the required solution

$$\frac{dt}{dx} + t\cos x = \sin x \cos x$$

This is of the form $\frac{dt}{dx} + Pt = Q$ where we have,

$$P = \cos x,\ Q = \sin x \cos x$$

Solution is given by $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $$t\,e^{\sin x} = \int \sin x\ \cos x\ e^{\sin x}\,dx + c$$

Put $\sin x = u \quad \therefore \quad \cos x\ dx = du$

$$t\,e^{\sin x} = \int u\,e^{u}\,du + c$$

ie., $$t\,e^{\sin x} = u\,e^{u} - e^{u} + c$$

or $$\sin y\,e^{\sin x} = e^{\sin x}(\sin x - 1) + c$$

>> Dividing the given equation by y^2 we have,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y\,x} = x$$

Put $\frac{1}{y} = t \quad \therefore \quad \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $\frac{-dt}{dx} + \frac{t}{x} = x$ or $\frac{dt}{dx} - \frac{t}{x} = -x$

This equation is a linear equation of the form $\frac{dt}{dx} + Pt = Q$, where

$$P = \frac{-1}{x} \text{ and } Q = -x$$

$$e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

The solution is $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $$t \cdot \frac{1}{x} = \int -x \cdot \frac{1}{x}\,dx + c$$

Thus $\frac{1}{xy} = -x + c$, is the required solution.

Dividing the given equation by y^3 we have

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{2}\left(1+\frac{1}{x}\right)\frac{1}{y^2} = \frac{-3}{x} \qquad \ldots (1)$$

Put $\quad \frac{1}{y^2} = t \quad \therefore \quad \frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx} \quad$ or $\quad \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dt}{dx}$

Hence (1) becomes $\frac{-1}{2}\frac{dt}{dx} - \frac{t}{2}\left(1+\frac{1}{x}\right) = \frac{-3}{x}$

or $\quad \frac{dt}{dx} + \left(1+\frac{1}{x}\right)t = \frac{6}{x}$

This equation is of the form $\frac{dt}{dx} + Pt = Q$ where

$$P = 1+\frac{1}{x} \text{ and } Q = \frac{6}{x}$$

$$e^{\int P\,dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\log x} = e^x \cdot x$$

The solution is $\; t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

i.e., $\quad t \cdot e^x x = \int \frac{6}{x}\,e^x x\;dx + c$

Thus $\quad \frac{x\,e^x}{y^2} = 6\,e^x + c,\;$ is the required solution.

Multiplying the given equation by y^2 we have

$$y^2\frac{dy}{dx} - y^3\tan x = \sin x\cos^2 x \qquad \ldots (1)$$

Put $\quad y^3 = t \quad \therefore \quad 3y^2\frac{dy}{dx} = \frac{dt}{dx} \quad$ or $\quad y^2\frac{dy}{dx} = \frac{1}{3}\frac{dt}{dx}$

Hence (1) becomes $\frac{1}{3}\frac{dt}{dx} - t\tan x = \sin x\cos^2 x$

or $\quad \dfrac{dt}{dx} - 3\tan x \cdot t = 3\sin x\cos^2 x$

This equation is of the form

$$\frac{dt}{dx} + Pt = Q, \text{ where } P = -3\tan x \text{ and } Q = 3\sin x\cos^2 x$$

$$\therefore \quad e^{\int P\,dx} = e^{\int -3\tan x\,dx} = e^{-3\log(\sec x)} = (\sec x)^{-3} = \cos^3 x$$

The solution is $\; t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad t\cos^3 x = \int 3\sin x\,\cos^2 x \cdot \cos^3 x\,dx + c$

ie., $\quad t\cos^3 x = 3\int \sin x\,\cos^5 x\,dx + c$

Put $\quad \cos x = u \quad \therefore \quad -\sin x\,dx = du$

$\therefore \quad t\cos^3 x = -3\int u^5\,du + c$

ie., $\quad t\cos^3 x = -\dfrac{u^6}{2} + c$

Thus $\quad (y\cos x)^3 = \dfrac{-\cos^6 x}{2} + c,\;$ is the required solution

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79. Solve : $r\sin\theta - \cos\theta\,\dfrac{dr}{d\theta} = r^2$

>> We have $\cos\theta\,\dfrac{dr}{d\theta} - r\sin\theta = -r^2$

Dividing by r^2 we get, $\dfrac{\cos\theta}{r^2}\,\dfrac{dr}{d\theta} - \dfrac{1}{r}\sin\theta = -1$

Put $\quad \dfrac{1}{r} = y\;$ and differentiate w.r.t θ

$$\frac{-1}{r^2}\frac{dr}{d\theta} = \frac{dy}{d\theta} \text{ and hence (1) becomes}$$

$$-\cos\theta\,\frac{dy}{d\theta} - y\sin\theta = -1 \quad \text{or} \quad \frac{dy}{d\theta} + (\tan\theta)\,y = \sec\theta$$

This is of the form, $\dfrac{dy}{d\theta} + Py = Q$ where

$$P = P(\theta) = \tan\theta \text{ and } Q = Q(\theta) = \sec\theta$$

Here $\quad e^{\int P\,d\theta} = e^{\int \tan\theta\,d\theta} = e^{\log(\sec\theta)} = \sec\theta$

Solution is given by $\; y\,e^{\int P\,d\theta} = \int Q\,e^{\int P\,d\theta}\,d\theta + c$

ie., $\quad y\sec\theta = \int \sec^2\theta\;d\theta + c$

ie., $\quad y\sec\theta = \tan\theta + c$

Thus $\quad \dfrac{\sec\theta}{r} = \tan\theta + c$, **is the required solution.**

80. *Solve :* $xy(1 + xy^2)\dfrac{dy}{dx} = 1$

Note that we cannot simplify the expression in respect of $\dfrac{dy}{dx}$

$\therefore$ Consider $\dfrac{dx}{dy} = xy + x^2y^3$ or $\dfrac{dx}{dy} - xy = x^2y^3$ Dividing by x^2 we get,

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3 \qquad \ldots (1)$$

Put $\quad \dfrac{1}{x} = t \quad \therefore \quad \dfrac{-1}{x^2}\dfrac{dx}{dy} = \dfrac{dt}{dy}$

Hence (1) becomes $\; -\dfrac{dt}{dy} - t\,y = y^3 \quad$ or $\quad \dfrac{dt}{dy} + t\,y = -y^3$

This equation is of the form $\dfrac{dt}{dy} + Pt = Q$, where

$$P = y \text{ and } Q = -y^3 \quad \therefore \quad e^{\int P\,dy} = e^{\int y\,dy} = e^{y^2/2}$$

The solution is $\; t\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $\quad t\,e^{y^2/2} = -\int y^3 e^{y^2/2}\,dy + c$

Put $\quad y^2/2 = u \quad \therefore \quad y\,dy = du$

Also $\quad y^2 \cdot y\;dy = y^2\;du \quad$ or $\quad y^3\;dy = 2u\;du$

$$t\,e^{y^2/2} = -2\int u\,e^u\,du + c$$

ie., $\quad t\,e^{y^2/2} = -2(u\,e^u - e^u) + c$, on integration by parts.

Thus $\quad \dfrac{e^{y^2/2}}{x} = 2\,e^{y^2/2}\left(1 - \dfrac{y^2}{2}\right) + c$, is the required solution

81. *Solve* : $(y \log x - 2)\, y\, dx = x\, dy$

\>> The given equation can be written in the form

$$\frac{dy}{dx} = \frac{y\,(y \log x - 2)}{x}$$

ie., $$\frac{dy}{dx} + \frac{2y}{x} = \frac{y^2 \log x}{x}$$ Dividing by y^2 we get,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{xy} = \frac{\log x}{x}$$

Put $$\frac{1}{y} = t \quad \therefore \quad \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

Hence (1) becomes $$-\frac{dt}{dx} + \frac{2t}{x} = \frac{\log x}{x}$$

ie., $$\frac{dt}{dx} - \frac{2t}{x} = -\frac{\log x}{x}$$

This equation is of the form $\frac{dt}{dx} + P\,t = Q$, where

$$P = \frac{-2}{x} \quad \text{and} \quad Q = -\frac{\log x}{x}$$

$$\therefore \quad e^{\int P\,dx} = e^{-\int \frac{2}{x} dx} = e^{-2\log x} = \frac{1}{x^2}$$

The solution is $$t\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$$

ie., $$\frac{t}{x^2} = -\int \log x \cdot \frac{1}{x^3}\, dx + c$$

ie., $$\frac{t}{x^2} = -\left\{\log x \cdot \frac{x^{-2}}{-2} - \int \frac{x^{-2}}{-2}\cdot\frac{1}{x}\, dx\right\} + c$$

ie., $$\frac{t}{x^2} = -\left[-\frac{\log x}{2x^2} + \frac{1}{2}\cdot\frac{x^{-2}}{-2}\right] + c$$

ie., $$\frac{t}{x^2} = \frac{1}{2x^2}\left(\log x + \frac{1}{2}\right) + c$$

Thus $$\frac{1}{x^2 y} = \frac{1}{2x^2}\left(\log x + \frac{1}{2}\right) + c,$$ is the required solution

82. *Solve*: $6y^2\,dx - x(x^3 + 2y)\,dy = 0$

The given equation can be written in the form

$$\frac{dx}{dy} = \frac{x(x^3 + 2y)}{6y^2}$$

ie., $\displaystyle \frac{dx}{dy} - \frac{x}{3y} = \frac{x^4}{6y^2}$ Dividing by x^4 we have,

$$\frac{1}{x^4}\frac{dx}{dy} - \frac{1}{3x^3 y} = \frac{1}{6y^2} \qquad \ldots (1)$$

Put $\displaystyle \frac{1}{x^3} = t \quad \therefore \quad \frac{-3}{x^4}\frac{dx}{dy} = \frac{dt}{dy}$ or $\displaystyle \frac{1}{x^4}\frac{dx}{dy} = \frac{-1}{3}\frac{dt}{dy}$

Hence (1) becomes $\displaystyle \frac{-1}{3}\frac{dt}{dy} - \frac{t}{3y} = \frac{1}{6y^2}$ or $\displaystyle \frac{dt}{dy} + \frac{t}{y} = \frac{-1}{2y^2}$

This equation is of the form $\displaystyle \frac{dt}{dy} + Pt = Q$, where

$$P = \frac{1}{y} \text{ and } Q = \frac{-1}{2y^2} \qquad \therefore \quad e^{\int P\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

The solution is $\displaystyle t\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $\displaystyle t\,y = \int \frac{-1}{2y^2}\cdot y\,dy + c$

ie., $\displaystyle t\,y = -\frac{\log y}{2} + c$

Thus $\displaystyle \frac{y}{x^3} = -\frac{\log y}{2} + c$, is the required solution.

83. *Solve*: $y(2xy + e^x)\,dx - e^x\,dy = 0$

>> The given equation can be written in the form,

$$\frac{dy}{dx} = \frac{y(2xy + e^x)}{e^x}$$

ie., $\displaystyle \frac{dy}{dx} - y = \frac{2xy^2}{e^x}$ Dividing by y^2 we get,

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \frac{2x}{e^x} \qquad \ldots (1)$$

Put $\displaystyle \frac{1}{y} = t \quad \therefore \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $-\frac{dt}{dx} - t = \frac{2x}{e^x}$ or $\frac{dt}{dx} + t = -\frac{2x}{e^x}$

This equation is of the form $\frac{dt}{dx} + Pt = Q$, where

$$P = 1 \text{ and } Q = \frac{-2x}{e^x} \quad \therefore \quad e^{\int P\,dx} = e^x$$

The solution is $t\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

ie., $\quad t\, e^x = \int \frac{-2x}{e^x} \cdot e^x\, dx + c$

ie., $\quad t\, e^x = -x^2 + c$

ie., $\quad \frac{e^x}{y} + x^2 = c,$ is the required solution.

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84. Solve : $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$

>> We have $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$. Dividing by $x^3 y^4$ we get

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{1}{x\,y^3} = -\frac{\cos x}{x^3}$$

Put $\frac{1}{y^3} = t \quad \therefore \quad \frac{-3}{y^4}\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{1}{y^4}\frac{dy}{dx} = \frac{-1}{3}\frac{dt}{dx}$

Hence (1) becomes, $-\frac{1}{3}\frac{dt}{dx} - \frac{t}{x} = -\frac{\cos x}{x^3}$

ie., $\quad \frac{dt}{dx} + 3\frac{t}{x} = \frac{3\cos x}{x^3}$

This equation is a linear equation of the form $\frac{dt}{dx} + Pt = Q$, where

$$P = \frac{3}{x} \text{ and } Q = \frac{3\cos x}{x^3} \quad \therefore \quad e^{\int P\,dx} = e^{\int (3/x)\,dx} = e^{3\log x} = x^3$$

Solution is given by $t\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

ie., $\quad t\, x^3 = \int \frac{3\cos x}{x^3} \cdot x^3\, dx + c$

ie., $\quad x^3/y^3 = 3\sin x + c,$ is the required solution.

EXERCISES

Solve the following differential equations

1. $\dfrac{dy}{dx} - \dfrac{y}{x+1} = e^{3x}(x+1)$

2. $\dfrac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x\,;\ y(\pi/2) = 0$

3. $\dfrac{dy}{dx} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}$

4. $x(x-1)\dfrac{dy}{dx} - y = \{x(x-1)\}^2$

5. $\sin 2x \dfrac{dy}{dx} - 2y = \tan x$

6. $(1+\sin x)\,dy + (x + y\cos x)\,dx = 0$ *(Solve by two methods)*

7. $xe^x \dfrac{dy}{dx} + e^x(1+x)\,y = 1$ *(Solve by two methods)*

8. $x\,dy + (y - x - xy\tan x)\,dx = 0$

9. $3y^2 \dfrac{dy}{dx} + 2xy^3 = 2xe^{-x^2}$

10. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2 y^6$

11. $\dfrac{dy}{dx} + \dfrac{y\log y}{x} = y\left(\dfrac{\log y}{x}\right)^2$

12. $\dfrac{dy}{dx} + \dfrac{2y}{3} = \dfrac{x}{y}$

13. $x^2\cos y\,dy - (1 + 3x\sin y)\,dy = 0$

14. $(y^4 - 2xy)\,dx + 3x^2\,dy = 0\ ;\ y(2) = 1$

15. $(x + \sqrt{xy})\,dy = y\,dx$ *(Solve by two methods)*

ANSWERS

1. $\dfrac{y}{x+1} = \dfrac{e^{3x}}{3} + c$

2. $y\sin x = 2x^2 - (\pi^2/2)$

3. $y\,e^{2\sqrt{x}} = 2\sqrt{x} + c$

4. $\dfrac{xy}{1-x} + \dfrac{x^3}{3} = c$

5. $y\cot x = \log\sqrt{\tan x} + c$

6. $y(1+\sin x) + \dfrac{x^2}{2} = c$

7. $xye^x = x + c$

8. $xy\cos x = \cos x + x\sin x + c$

9. $y^3 e^x = x^2 + c$

10. $\dfrac{1}{x^5 y^5} - \dfrac{5}{2x^2} = c$

11. $\dfrac{1}{x \log y} = \dfrac{1}{2x^2} + c$

12. $y^{3/2} e^x = \dfrac{3}{2}(x-1)e^x + c$

13. $4x \sin y + 1 = cx^4$

14. $x^2 = y^3 (x+2)$

15. $\sqrt{x} = \sqrt{y}\,(\log \sqrt{y} + c)$

* *Refer articles 6.5 & 6.6 for Methods at a glance and Type recognition*

6.4 | Orthogonal Trajectories

Introduction

Basically we know that, two curves intersect each other orthogonally if the tang at the point of intersection are at right angles. Further we know from differe calculus that, for a cartesian curve $y = f(x)$, $m = \dfrac{dy}{dx}$ represents the slope o tangent.

In order to show that two curves intersect orthogonally we simply obtain $\dfrac{dy}{dx}$ fo two curves say m_1, m_2 and establish $m_1 m_2 = -1$ being the condition for two lin be perpendicular. In fact the orthogonality of two polar curves also has been discu in chapter - 2 (*article* 2.2).

With the knowledge of differential equations, given a family of curves it is possib determine another family of curves which intersects each member of the given fa orthogonally and we discuss this concept in detail.

6.41 Orthogonal trajectories of cartesian and polar family of curves

An equation of the form $f(x, y, c) = 0$ where c is a fixed constant represents a cu

For example $x^2 + y^2 = 4$ is a circle, $y^2 = 4x$ is a parabola etc.

On the otherhand if c is an arbitrary constant (c *is a parameter*) the equa $f(x, y) = c$ represents a one parameter family of curves. For each value of c we different curves of the same family.

For example $x^2 + y^2 = r^2$ (*r is arbitrary*) represents a family of concentric circles.

Definition If two family of curves are such that every member of one family inters every member of the other family at right angles then they are said to be *orthog trajectories* of each other.

Method of finding the orthogonal trajectories

Case - (i) Cartesian family $f(x, y, c) = 0$

We differentiate *w.r.t.* x and eliminate the parameter c. The equation so obtained is called as the differential equation of the given family.

We know that if $\tan\psi = \dfrac{dy}{dx}$ is the slope of a given line then the slope of the line perpendicular to it is $-\dfrac{1}{\tan\psi} = -\dfrac{dx}{dy}$. Accordingly in the differential equation of the given family we shall replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ to arrive at a new differential equation. Solving this new differential equation we get the orthogonal trajectories of the given family of curves.

Self orthogonal family : If the differential equation of the given family remains unaltered after replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ then the given family of curves is said to be *self orthogonal*.

Case - (ii) : Polar family $f(r, \theta, c) = 0$

We know that $\tan\phi = r\dfrac{d\theta}{dr}$ for a polar curve where ϕ is the angle between the radius vector and the tangent. $\phi_2 - \phi_1 = 90^0$ is the condition for two polar curves to be orthogonal.

$$\phi_2 = 90^0 + \phi_1 \Rightarrow \tan\phi_2 = \tan(90^0 + \phi_1)$$

ie., $\quad \tan\phi_2 = -\cot\phi_1 \quad$ or $\quad \tan\phi_2 = \dfrac{-1}{\tan\phi_1}$

But $\tan\phi_1 = r\dfrac{d\theta}{dr}$ for the given curve and $\tan\phi_2 = r\dfrac{d\theta}{dr}$ for the orthogonal curve at the same point.

$$\therefore \quad r\frac{d\theta}{dr} \text{ for the curve to be replaced by } \frac{-1}{r\dfrac{d\theta}{dr}}$$

ie., $\quad -r^2\dfrac{d\theta}{dr}$ to be replaced by $\dfrac{dr}{d\theta}$ or vice - versa.

In otherwords, we have to differentiate $f(r, \theta, c) = 0$ *w.r.t.* θ and eliminate c to obtain the D.E of the given family. We have to replace $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ to obtain the new D.E and solve the same to obtain the required orthogonal trajectories.

Working procedure for problems

Case - i : (Cartesian family)

- Given $f(x, y, c) = 0$, differentiate *w.r.t* x and eliminate c.
- Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ and solve the equation.

Case - ii : (Polar family)

- Given an equation in r and θ, we prefer to take logarithms first and then differentiate *w.r.t* θ
- After ensuring that the given parameter is eliminated we replace $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ and solve the equation.

[illegible]

Let us find the O.T of the family of circles $x^2 + y^2 = a^2$.

Differentiating *w.r.t* x we get, $2x + 2y\dfrac{dy}{dx} = 0$ or $x + y\dfrac{dy}{dx} = 0$

This is the D.E of the given family.

Now, let us replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$\therefore \quad x + y\left(-\dfrac{dx}{dy}\right) = 0$ is the D.E of the orthogonal family.

ie., $\quad x\,dy - y\,dx = 0$. Dividing by $x\,y$ we get

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \quad \int \frac{dy}{y} - \int \frac{dx}{x} = c$$

ie., $\quad \log y - \log x = c \quad$ or $\quad \log(y/x) = \log k \quad$ *(say)*

ie., $\quad y/x = k \quad$ or $\quad y = kx$, is the required O.T.

Geometrically $y = kx$ for different values of k represent a family of straight lines passing through the origin. Let us draw a circle with centre origin and a straight line passing through the origin. (Figure-1)

Also, let us draw a family of circles and a family of straight lines of the same category (Figure - 2)

It may be observed that in Figure - 1 the two curves, circle and the straight line intersect on the circumference of the circle and a tangent is drawn at the point of intersection The angle of intersection can easily be seen as 90^0. In Figure - 2 we observe that any straight line passing through the origin intersects every member of the family of circles at right angles and vice-versa.

Hence, theoretically and geometrically we can say that the family of circles ($x^2 + y^2 = a^2$) and the family of straight lines passing through the origin ($y = kx$) are orthogonal trajectories of each other.

WORKED PROBLEMS

Find the O.T of the family of parabolas

>> Consider $\dfrac{y^2}{x} = 4a$ . . . (1)

(If the parameter is on one side of the equation exclusively, then the same gets eliminated once we differentiate)

Now differentiating (1) $w.r.t\ x$ we have

$$\frac{x \cdot 2y\,\dfrac{dy}{dx} - y^2 \cdot 1}{x^2} = 0 \quad \text{or} \quad 2xy\,\frac{dy}{dx} - y^2 = 0$$

ie., $2x\,\dfrac{dy}{dx} - y = 0,$ is the D.E of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy}\right) - y = 0 \qquad \text{or} \qquad 2x\,dx + y\,dy = 0$$

$$\Rightarrow \qquad \int 2x\ dx + \int y\,dy = c$$

ie., $\quad x^2 + \dfrac{y^2}{2} = c \quad \text{or} \quad 2x^2 + y^2 = 2c = k \quad (say)$

Thus $\quad \mathbf{2x^2 + y^2 = k}$, is the required O.T.

86. *Find the O.T of the family of astroids* $x^{2/3} + y^{2/3} = a^{2/3}$

>> Consider $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating *w.r.t x,* we have

$$\frac{2}{3} \cdot x^{-1/3} + \frac{2}{3} \cdot y^{-1/3} \frac{dy}{dx} = 0$$

ie., $\quad x^{-1/3} + y^{-1/3} \dfrac{dy}{dx} = 0$, is the D.E of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$x^{-1/3} + y^{-1/3} \left(-\frac{dx}{dy} \right) = 0 \quad ie., \; x^{-1/3}\, dy = y^{-1/3}\, dx$$

ie., $\quad y^{1/3}\, dy = x^{1/3}\, dx$ by separating the variables.

$\Rightarrow \quad \int y^{1/3}\, dy - \int x^{1/3}\, dx = c$

ie., $\quad \dfrac{y^{4/3}}{(4/3)} - \dfrac{x^{4/3}}{(4/3)} = c \quad \text{or} \quad x^{4/3} - y^{4/3} = -\dfrac{4c}{3} = k \quad (say)$

Thus $\quad x^{4/3} - y^{4/3} = k$ is the required O.T

87. *Find the O.T of the family* $y^2 = cx^3$

>> We have $\dfrac{y^2}{x^3} = c$

Differentiating *w.r.t x* we have,

$$\frac{x^3 \cdot 2y \dfrac{dy}{dx} - y^2 \cdot 3x^2}{x^6} = 0 \quad \text{or} \quad 2x^3 y \frac{dy}{dx} = 3x^2 y^2$$

ie., $\quad 2x \dfrac{dy}{dx} = 3y$ Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy} \right) = 3y$$

ie., $\quad 2x\, dx + 3y\, dy = 0$

$\Rightarrow \qquad \int 2x\,dx + \int 3y\,dy = c$

ie., $\qquad x^2 + \dfrac{3y^2}{2} = c \quad \text{or} \quad 2x^2 + 3y^2 = 2c = k \quad (say)$

Thus $\quad \mathbf{2x^2 + 3y^2 = k}$, **is the required O.T.**

88. *Show that the family of curves $x^3 - 3xy^2 = c_1$ and $y^3 - 3x^2y = c_2$ are O.T of each other.*

\>> Let us consider $x^3 - 3xy^2 = c_1$ and differentiate *w.r.t* x.

$$3x^2 - 3\left(x \cdot 2y\frac{dy}{dx} + y^2\right) = 0$$

ie., $\qquad x^2 - y^2 = 2xy\dfrac{dy}{dx}$, is the D.E of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$x^2 - y^2 = 2xy\left(-\frac{dx}{dy}\right) \quad \text{or} \quad 2xy\,dx + (x^2 - y^2)\,dy = 0$$

(This is a homogeneous equation. But it is also exact)

Let $\qquad M = 2xy \quad \text{and} \quad N = x^2 - y^2$

$\dfrac{\partial M}{\partial y} = 2x$ and $\dfrac{\partial N}{\partial x} = 2x$. Hence the equation is exact.

The solution is given by $\quad \int M\,dx + \int N(y)\,dy = c$

ie., $\qquad \int 2xy\,dx + \int -y^2\,dy = c$

ie., $\qquad x^2y - \dfrac{y^3}{3} = c \quad \text{or} \quad 3x^2y - y^3 = 3c$

$$y^3 - 3x^2y = c_2 \ (say) \text{ is the required O.T where } c_2 = -3c$$

Thus $x^3 - 3xy^2 = c_1$ and $y^3 - 3x^2y = c_2$ are orthogonal trajectories of each other.

>> We shall draw a circle of the given family so as to write its equation

Centre $(a, 0)$ and radius is 'a' since the circle passes through the origin

$\therefore$ Equation of the given family of circles is

$$(x-a)^2+(y-0)^2 = a^2$$

ie., $$x^2-2ax+y^2 = 0 \qquad \ldots(1)$$

Differentiating w.r.t x we have,

$$2x-2a+2yy_1 = 0 \quad \text{where} \quad y_1 = \frac{dy}{dx}$$

$\therefore$ $a = x+yy_1$ and substituting this value of 'a' in (1) we get,

$$x^2-2(x+yy_1)x+y^2 = 0$$

ie., $y^2-x^2 = 2xyy_1$ is the D.E of the given family.

Now let us replace $y_1 = \dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$$y^2-x^2 = 2xy\left(-\frac{dx}{dy}\right)$$

or $$\frac{dx}{dy} = \frac{x^2-y^2}{2xy}$$ which is a homogeneous equation.

Put $x = vy \quad \therefore \quad \dfrac{dx}{dy} = v+y\dfrac{dv}{dy}$

Hence $$v+y\frac{dv}{dy} = y^2\frac{(v^2-1)}{2vy^2}$$

ie., $\quad y\dfrac{dv}{dy} = \dfrac{v^2-1}{2v} - v \quad$ or $\quad y\dfrac{dv}{dy} = -\dfrac{(1+v^2)}{2v}$

$\therefore \quad \dfrac{2v\,dv}{1+v^2} = -\dfrac{dy}{y}$ by separating the variables.

$$\Rightarrow \quad \int \frac{2v}{1+v^2}\,dv + \int \frac{dy}{y} = c$$

ie., $\quad \log(1+v^2) + \log y = c$

ie., $\quad \log\left[(1+v^2)\,y\right] = \log k$ (say) where $v = x/y$

$$\left(1+\frac{x^2}{y^2}\right)y = k \quad \text{or} \quad x^2+y^2 = ky$$

ie., $\quad x^2+y^2-ky = 0 \quad$ and $k = 2b$ for convenience.

$\therefore \quad x^2+y^2-2by = 0 \quad$ or $\quad x^2+(y-b)^2-b^2 = 0$

ie., $\quad x^2+(y-b)^2 = b^2$

This is the equation of the family of circles passing through the origin having their centres on the y - axis.

Thus we have proved the required result.

90. Find the orthogonal trajectories of the famil [illegible]

>> We have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1 \qquad \ldots (1)$

Differentiating $w.r.t\ x$ we have,

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0, \quad \text{where } y_1 = \frac{dy}{dx}$$

ie., $\quad \dfrac{x}{a^2} = \dfrac{-yy_1}{b^2+\lambda} \qquad \ldots (2)$

Also from (1) $\quad \dfrac{x^2}{a^2} - 1 = \dfrac{-y^2}{b^2+\lambda} \quad$ or $\quad \dfrac{x^2-a^2}{a^2} = \dfrac{-y^2}{b^2+\lambda} \qquad \ldots (3)$

Now, dividing (2) by (3) we get,

$$\frac{x}{x^2-a^2} = \frac{yy_1}{y^2} \quad \text{or} \quad \frac{x}{x^2-a^2} = \frac{y_1}{y}$$

Now let us replace $y_1 = \frac{dy}{dx}$ by $-\frac{dx}{dy}$

$\therefore \quad \frac{x}{x^2 - a^2} = \frac{1}{y}\left(-\frac{dx}{dy}\right)$ or $y\,dy = -\frac{(x^2 - a^2)}{x}\,dx$ by separating the variables

$\Rightarrow \quad \int y\,dy = -\int x\,dx + a^2 \int \frac{dx}{x} + c$

ie., $\quad \frac{y^2}{2} = \frac{-x^2}{2} + a^2 \log x + c$

Thus $x^2 + y^2 - 2a^2 \log x - b = 0$ where $b = 2c$, is the required orthogonal trajectories.

91. *Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2$ λ being the parameter*

OR

Show that the orthogonal trajectories of a family of coaxial circles $x^2 + y^2 + 2\lambda x + c = 0$ λ being the parameter is also a system of coaxial circles

>> Consider $x^2 + y^2 + 2\lambda x + c = 0$ ... (1)

Differentiating *w.r.t* x we have,

$$2x + 2y\,y_1 + 2\lambda = 0, \quad \text{where } y_1 = \frac{dy}{dx}$$

$\lambda = -(x + y\,y_1)$ and substituting this value in (1) we get,

$$(x^2 + y^2) - 2(x + yy_1)x + c = 0$$

ie., $\quad y^2 - x^2 - 2xy\,y_1 + c = 0 \quad$ or $\quad -2xy\,y_1 = x^2 - y^2 - c$

$$-y_1 = \frac{x^2 - y^2 - c}{2xy}$$

Now replacing $y_1 = \frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have

$$\frac{dx}{dy} = \frac{x^2 - y^2 - c}{2xy} \quad \text{or} \quad \frac{dx}{dy} = \frac{x}{2y} - \frac{(y^2 + c)}{2xy}$$

ie., $\quad 2x\frac{dx}{dy} - \frac{x^2}{y} = -\left(y + \frac{c}{y}\right)$ ... (2)

Now put $x^2 = t \quad \therefore \quad 2x\dfrac{dx}{dy} = \dfrac{dt}{dy}$

Hence (2) becomes $\dfrac{dt}{dy} - \dfrac{t}{y} = -\left(y + \dfrac{c}{y}\right)$

This is a linear equation of the form

$$\frac{dt}{dy} + Pt = Q, \text{ where } P = \frac{-1}{y} \text{ and } Q = -\left(y + \frac{c}{y}\right)$$

Hence $e^{\int P\,dy} = e^{\int \frac{-1}{y}dy} = e^{-\log y} = \dfrac{1}{y}$

The solution is $t\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c_1$

ie., $$\frac{t}{y} = \int -\left(y + \frac{c}{y}\right)\frac{1}{y}\,dy + c_1$$

ie., $$\frac{t}{y} = -\int dy - c\int \frac{1}{y^2}\,dy + c_1$$

ie., $$\frac{x^2}{y} = -y + \frac{c}{y} + c_1 \quad \text{or} \quad x^2 = -y^2 + c + c_1 y$$

$$x^2 + y^2 - c_1 y - c = 0 \quad ; \quad \text{Let } -c_1 = 2\mu \text{ and } -c = c_2$$

Then $x^2 + y^2 + 2\mu y + c_2 = 0$ (*μ is the parameter*) is the required orthogonal trajectories which is also a coaxial system of circles.

92. *If $u(x, y)$ and $v(x, y)$ are two functions satisfying the condition*

[illegible]

show that the families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are arbitrary constants are orthogonal trajectories of each other.

Sol. It is enough if we show that the product of the slope of the tangents of the two family of curves at the point of intersection is equal to -1.

Consider $u(x, y) = c_1$ and differentiate *w.r.t.* x treating y as a function of x. This will give us

$$u_x + u_y\frac{dy}{dx} = 0, \text{ where } u_x = \frac{\partial u}{\partial x} \text{ and } u_y = \frac{\partial u}{\partial y}$$

$$\frac{dy}{dx} = -\frac{u_x}{u_y} = m_1 \text{ (say)}$$

Similarly for the family $v(x, y) = c_2$ we can obtain

$$\frac{dy}{dx} = -\frac{v_x}{v_y} = m_2 \quad (say)$$

Now consider the product of the slopes of the tangents which being m_1 [illegible] m_2

$$m_1 \cdot m_2 = -\frac{u_x}{u_y} \cdot -\frac{v_x}{v_y} \text{ [illegible]}$$

But, we have by data $u_x = v_y$ and $v_x = -u_y$

Hence $\quad m_1 \cdot m_2 = \frac{v_y \cdot -u_y}{u_y\, v_y} = -1$

This proves the required result.

93. *Show that the family of parabolas $y^2 = 4a(x+a)$ is self orthogonal*

>> Consider $\quad y^2 = 4a(x+a)$ $\quad \dots (1)$

Differentiating *w.r.t* x, we have

$$2y\frac{dy}{dx} = 4a \quad \therefore \quad a = \frac{y\,y_1}{2} \quad \text{where} \quad y_1 = \frac{dy}{dx}$$

Substituting this value of 'a' in (1) we have,

$$y^2 = 2\,y\,y_1\left(x + \text{[illegible]}\right), \quad \text{or} \quad y = 2\,x\,y_1 + \text{[illegible]}$$

Thus we have, $\quad y = 2\,x\,y_1 + y\,y_1^2$ $\quad \dots (2)$

This is the D.E of the given family.

Now replacing y_1 by $-1/y_1$, (2) becomes

[illegible]

$$y\,y_1^2 + 2\,x\,y_1 = y \quad \dots (3)$$

(3) the D.E of the orthogonal family which is same as (2) being the D.E of the given family

Thus the family of parabolas y^2 [illegible] $4a(x+a)$ is self orthogonal.

94. *Find the orthogonal trajectories of the family of curves* [illegible] $+$ [illegible] $= 1$, *where* λ *is* [illegible]

Show that the family of conics [illegible] $+$ [illegible] $= 1$, *where* λ *is a parameter is self orthogonal*

>> Consider $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$ ... (1)

Differentiating $w.r.t\ x$ we have,

[illegible] or $\dfrac{x}{a^2+\lambda}=\dfrac{-yy_1}{b^2+\lambda}$ ... (2)

We have to eliminate λ to obtain the D.E. of the given family.

We have a property in ratio and proportion that if [illegible], then it is also equal to $\dfrac{a-c}{b-d}$. That is $\dfrac{a}{b}=\dfrac{c}{d}$ Also $\dfrac{a-c}{b-d}$

Hence (2) becomes

[illegible] $\dfrac{x+yy_1}{[illegible]}$

$\Rightarrow \quad \dfrac{x}{a^2+\lambda}=\dfrac{x+yy_1}{a^2-b^2}$ and $\dfrac{-yy_1}{b^2+\lambda}=\dfrac{x+yy_1}{a^2-b^2}$

ie., [illegible] ... (3)

and $\dfrac{y}{b^2+\lambda}=\dfrac{x+yy_1}{-y_1(a^2-b^2)}$ ... (4)

Now $(3)\times x+(4)\times y$ will give us

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=\frac{x(x+yy_1)}{(a^2-b^2)}+\frac{y(x+yy_1)}{-y_1(a^2-b^2)}$$

Using (1) in the L.H.S we have,

$$1 = \frac{x + y\,y_1}{a^2 - b^2}\left(x - \frac{y}{y_1}\right) \quad \text{or} \quad (x + y\,y_1)\left(x - \frac{y}{y_1}\right) = a^2 - b^2 \qquad \dots (5)$$

This is the D.E of the given family.

Now replacing y_1 by $-1/y_1$ in (5) we get the D E of the orthogonal family

ie., $$\left(x - \frac{y}{y_1}\right)(x + y\,y_1) = (a^2 - b^2) \qquad \dots (6)$$

It may be observed that (5) and (6) are identically equal. In otherwords the D E of the given family and the orthogonal family are the same

Hence the given family of conics is self orthogonal.

>> $$y = k\,e^{-2x} + 3x \qquad \dots (1)$$

$$\frac{dy}{dx} = -2ke^{-2x} + 3$$

ie., $$\frac{dy}{dx} = -2(y - 3x) + 3, \text{ by using (1) } [\,k \textit{ is eliminated}\,]$$

This is the *d e* of the given family and now replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$-\frac{dx}{dy} = -2(y - 3x) + 3 \quad \text{or} \quad \frac{dx}{dy} = 2y - (6x + 3)$$

is the *d e* of the orthogonal family which has to be solved

We have $\frac{dx}{dy} + 6x = (2y - 3)$ which is of the form

$$\frac{dx}{dy} + Px = Q\,; \text{ where } P = 6 \text{ and } Q = (2y - 3)$$

Solution : $x\,e^{\int P\,dy} = c + \int Q\,e^{\int P\,dy}\,dy$

ie., $$x\,e^{6y} = c + \int (2y - 3)\,e^{6y}\,dy$$

ie., $$x\,e^{6y} = c + \left\{ (2y - 3)\,\frac{e^{6y}}{6} - (2)\,\frac{e^{6y}}{36} \right\}, \text{ by the product rule.}$$

ie., $\quad x\,e^{6y} = c + \left\{ (2y-3)\,\dfrac{e^{6y}}{6} - \dfrac{e^{6y}}{18} \right\}$ $\quad \dots (2)$

This is the O.T of the given family. Further to find a member of this family passing through the point (0 , 3) we have from (2),

$$0 = c + \left\{ 3\,\frac{e^{18}}{6} - \frac{e^{18}}{18} \right\}$$

ie., $\quad -c = e^{18}\left(\dfrac{1}{2} - \dfrac{1}{18}\right)$ or $-c = \dfrac{8}{18}\,e^{18} \quad \therefore \quad c = -\dfrac{4}{9}\,e^{18}$

Using this value of c in (2) we have,

$$x\,e^{6y} = -\frac{4}{9}\,e^{18} + \left\{ (2y-3)\,\frac{e^{6y}}{6} - \frac{e^{6y}}{18} \right\}$$

or $\quad 18\,x\,e^{6y} = -8\,e^{18} + e^{6y}\,(6y-10)$

Thus $9x\,e^{6y} = -4\,e^{18} + (3y-5)\,e^{6y}$ is the member of the orthogonal family passing through the point (0 , 3)

[illegible]

>> We have $r = a\,(1+\sin\theta)$

$\Rightarrow \quad \log r = \log a + \log(1+\sin\theta)$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\,\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\,\dfrac{d\theta}{dr}$ we get,

$$\frac{1}{r}\left(-r^2\,\frac{d\theta}{dr}\right) = \frac{\cos\theta}{1+\sin\theta}$$

ie., $\quad -r\,\dfrac{d\theta}{dr} = \dfrac{\cos\theta}{1+\sin\theta}$ or $\dfrac{1+\sin\theta}{\cos\theta}\,d\theta = -\dfrac{dr}{r}$ by separating the variables

Hence $\displaystyle\int \frac{dr}{r} + \int \frac{1+\sin\theta}{\cos\theta}\,d\theta = c$

ie., $\quad \log r + \displaystyle\int \sec\theta\,d\theta + \int \tan\theta\,d\theta = c$

ie., $\quad \log r + \log(\sec\theta + \tan\theta) + \log(\sec\theta) = c$

ie., $\quad \log\left[\, r\,(\sec\theta + \tan\theta)\,\sec\theta \,\right] = \log b$ *(say)*

Using (1) in the L.H.S we have,

$$1 = \frac{x + y\,y_1}{a^2 - b^2}\left(x - \frac{y}{y_1}\right) \quad \text{or} \quad (x + y\,y_1)\left(x - \frac{y}{y_1}\right) = a^2 - b^2 \qquad \ldots (5)$$

This is the D.E of the given family.

Now replacing y_1 by $-1/y_1$ in (5) we get the D.E of the orthogonal family

ie., $$\left(x - \frac{y}{y_1}\right)(x + y\,y_1) = (a^2 - b^2) \qquad \ldots (6)$$

It may be observed that (5) and (6) are identically equal. In otherwords the D.E of the given family and the orthogonal family are the same.

Hence the given family of conics is self orthogonal.

\>> $$y = k\,e^{-2x} + 3x \qquad \ldots (1)$$

$$\frac{dy}{dx} = -2ke^{-2x} + 3$$

ie., $$\frac{dy}{dx} = -2(y - 3x) + 3, \text{ by using (1). [} k \text{ is eliminated]}$$

This is the *d.e* of the given family and now replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$-\frac{dx}{dy} = -2(y - 3x) + 3 \quad \text{or} \quad \frac{dx}{dy} = 2y - (6x + 3)$$

is the *d.e* of the orthogonal family which has to be solved.

We have $\frac{dx}{dy} + 6x = (2y - 3)$ which is of the form

$$\frac{dx}{dy} + Px = Q; \text{ where } P = 6 \text{ and } Q = (2y - 3)$$

Solution: $x\,e^{\int P\,dy} = c + \int Q\,e^{\int P\,dy}\,dy$

ie., $$x\,e^{6y} = c + \int (2y - 3)\,e^{6y}\,dy$$

ie., $$x\,e^{6y} = c + \left\{(2y - 3)\frac{e^{6y}}{6} - (2)\frac{e^{6y}}{36}\right\}, \text{ by the product rule.}$$

ie., $$x\,e^{6y} = c + \left\{ (2y-3)\,\frac{e^{6y}}{6} - \frac{e^{6y}}{18} \right\} \qquad \ldots (2)$$

This is the O T of the given family Further to find a member of this family passing through the point (0 , 3) we have from (2),

$$0 = c + \left\{ 3\,\frac{e^{18}}{6} - \frac{e^{18}}{18} \right\}$$

ie., $$-c = e^{18}\left(\frac{1}{2} - \frac{1}{18}\right) \quad \text{or} \quad -c = \frac{8}{18}\,e^{18} \quad \therefore \quad c = -\frac{4}{9}\,e^{18}$$

Using this value of c in (2) we have,

$$x\,e^{6y} = -\frac{4}{9}\,e^{18} + \left\{ (2y-3)\,\frac{e^{6y}}{6} - \frac{e^{6y}}{18} \right\}$$

or $$18\,x\,e^{6y} = -8\,e^{18} + e^{6y}\,(6y-10)$$

Thus $9x\,e^{6y} = -4\,e^{18} + (3y-5)\,e^{6y}$ is the member of the orthogonal family passing through the point (0 , 3)

96 [illegible]

>> We have $r = a\,(1+\sin\theta)$

$\Rightarrow$ $$\log r = \log a + \log(1+\sin\theta)$$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\,\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\,\dfrac{d\theta}{dr}$ we get,

$$\frac{1}{r}\left(-r^2\,\frac{d\theta}{dr}\right) = \frac{\cos\theta}{1+\sin\theta}$$

ie., $$-r\,\frac{d\theta}{dr} = \frac{\cos\theta}{1+\sin\theta} \quad \text{or} \quad \frac{1+\sin\theta}{\cos\theta}\,d\theta = -\frac{dr}{r}$$ by separating the variables

Hence $$\int \frac{dr}{r} + \int \frac{1+\sin\theta}{\cos\theta}\,d\theta = c$$

ie., $$\log r + \int \sec\theta\,d\theta + \int \tan\theta\,d\theta = c$$

ie., $$\log r + \log(\sec\theta + \tan\theta) + \log(\sec\theta) = c$$

ie., $$\log\Big[\,r\,(\sec\theta + \tan\theta)\,\sec\theta\,\Big] = \log b \ (say)$$

$$\Rightarrow \quad r\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)\frac{1}{\cos\theta}=b$$

ie., $$\frac{r(1+\sin\theta)}{\cos^2\theta}=b \quad \text{or} \quad \frac{r(1+\sin\theta)}{(1-\sin^2\theta)}=b$$

Thus $r = b(1-\sin\theta)$ **is the required O.T.**

97. *Find the O.T of the family* $2a/r = 1-\cos\theta$

>> We have $2a/r = 1-\cos\theta$

$\Rightarrow$ $\log 2a - \log r = \log(1-\cos\theta)$

Differentiating *w.r.t* θ we have,

$$\frac{-1}{r}\frac{dr}{d\theta}=\frac{\sin\theta}{1-\cos\theta}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ and simplifying R.H.S we have

$$\frac{-1}{r}\left(-r^2\frac{d\theta}{dr}\right)=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

ie., $$r\frac{d\theta}{dr}=\cot(\theta/2) \quad \text{or} \quad \tan(\theta/2)\,d\theta=\frac{dr}{r}$$

$$\int\frac{dr}{r}-\int\tan(\theta/2)\,d\theta=c$$

ie., $$\log r-\frac{\log\sec(\theta/2)}{(1/2)}=c$$

ie., $$\log\left[r/\sec^2(\theta/2)\right]=\log b \quad (say)$$

$$\Rightarrow \quad r/\sec^2(\theta/2)=b \quad \text{or} \quad r\cos^2(\theta/2)=b$$

This is the required equation of the O.T which can be put in the following form

$$r\cdot\frac{1}{2}(1+\cos\theta)=b \quad \text{or} \quad r(1+\cos\theta)=2b$$

Thus $2b/r = 1+\cos\theta$ **is the required O.T.**

98. *Find the O.T. of the family* $r^n \cos n\theta = a^n$

>> We have $r^n \cos n\theta = a^n$

$\Rightarrow \quad n \log r + \log(\cos n\theta) = n \log a$

Differentiating *w.r.t* θ we have

$$\frac{n}{r}\frac{dr}{d\theta} + \left(\frac{-n \sin n\theta}{\cos n\theta}\right) = 0 \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \tan n\theta \quad \text{or} \quad -r\frac{d\theta}{dr} = \tan n\theta$$

$$\frac{d\theta}{\tan n\theta} = \frac{dr}{-r} \quad \text{by separating the variables.}$$

$$\Rightarrow \quad \int \frac{dr}{r} + \int \cot n\theta \, d\theta = c$$

ie., $\quad \log r + \dfrac{1}{n}\log(\sin n\theta) = c \quad$ or $\quad n \log r + \log(\sin n\theta) = nc$

ie., $\quad \log(r^n \sin n\theta) = \log b$ (*say*)

Thus $r^n \sin n\theta = b$, **is the required O.T.**

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99. *Find the orthogonal trajectories of the family of curves* $r^n = a^n \cos n\theta$

>> $r^n = a^n \cos n\theta$

$\Rightarrow \quad n \log r = n \log a + \log(\cos n\theta)$

Differentiating w.r.t θ we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan n\theta$$

ie., $\quad r\dfrac{d\theta}{dr} = \tan n\theta$

$$\frac{d\theta}{\tan n\theta} = \frac{dr}{r} \quad \Rightarrow \quad \int \frac{dr}{r} - \int \cot n\theta \, d\theta = c$$

ie., $\log r = \frac{1}{n}\log(\sin n\theta) + c$ or $n\log r = \log(\sin n\theta) + nc$

ie., $\log\left[\frac{r^n}{\sin n\theta}\right] = \log k\ (say) \rightarrow r^n = k\sin n\theta$

Thus $\boldsymbol{r^n = k\sin n\theta}$, is the required orthogonal trajectory

100. *Show that the orthogonal trajectories of the family of cardioides $r = a\cos^2(\theta/2)$ is another family of cardioides $r = b\sin^2(\theta/2)$*

\>> We have $r = a\cos^2(\theta/2)$

$\Rightarrow \quad \log r = \log a + 2\log\cos(\theta/2)$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 2\cdot\frac{1}{\cos(\theta/2)}\cdot -\sin(\theta/2)\cdot\frac{1}{2}$$

ie., $\frac{1}{r}\frac{dr}{d\theta} = -\tan(\theta/2)$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan(\theta/2)$$

ie., $-r\frac{d\theta}{dr} = -\tan(\theta/2)$ or $\cot(\theta/2)\,d\theta = \frac{dr}{r}$ by separating the variables.

$\therefore \quad \int\frac{dr}{r} - \int\cot(\theta/2)\,d\theta = c$

ie., $\log r - 2\log\sin(\theta/2) = c$

ie., $\log\left(\frac{r}{\sin^2(\theta/2)}\right) = \log b\ (say)$

Thus $\boldsymbol{r = b\sin^2(\theta/2)}$ **is the required O.T..**

101. *Find the orthogonal trajectories of the family of curves $r = 4a\sec\theta\tan\theta$*

\>> We have $r = 4a\sec\theta\tan\theta$

$\Rightarrow \quad \log r = \log(4a) + \log(\sec\theta) + \log(\tan\theta)$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sec\theta\tan\theta}{\sec\theta} + \frac{\sec^2\theta}{\tan\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = \frac{\tan^2\theta + \sec^2\theta}{\tan\theta}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\tan^2\theta + \sec^2\theta}{\tan\theta}$$

or $$\frac{\tan\theta}{\tan^2\theta + \sec^2\theta}\,d\theta = -\frac{dr}{r}$$ by separating the variables

$$\Rightarrow \quad \int\frac{dr}{r} + \int\frac{\tan\theta}{\tan^2\theta + \sec^2\theta}\,d\theta = c$$

ie., $$\log r + \int\frac{\sin\theta/\cos\theta}{\sin^2\theta/\cos^2\theta + 1/\cos^2\theta}\,d\theta = c$$

ie., $$\log r + \int\frac{\sin\theta\cos\theta}{\sin^2\theta + 1}\,d\theta = c$$

ie., $$\log r + \frac{1}{2}\log(1 + \sin^2\theta) = c$$

ie., $$\log\left[r\sqrt{1+\sin^2\theta}\right] = \log b \quad (say) \quad \Rightarrow \quad r\sqrt{1+\sin^2\theta} = b$$

Thus $\boldsymbol{r^2(1+\sin^2\theta) = b^2}$, is the required O.T.

102. *Find the orthogonal trajectories of the family of Lemniscates $r^2 = a^2\cos 2\theta$*

>> We have, $r^2 = a^2\cos 2\theta$

$$\Rightarrow \quad 2\log r = 2\log a + \log(\cos 2\theta)$$

Differentiating *w.r.t* θ we get,

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta} \quad or \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we get,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan 2\theta \quad \text{or} \quad r\frac{d\theta}{dr} = \tan 2\theta$$

$$\frac{d\theta}{\tan 2\theta} = \frac{dr}{r},$$ by separating the variables.

$\Rightarrow \quad \int \frac{dr}{r} - \int \cot 2\theta \, d\theta = c$

ie., $\quad \log r - \frac{1}{2} \log(\sin 2\theta) = c$

ie., $\quad \log \left| \frac{r}{\sqrt{\sin 2\theta}} \right| = \log b$ *(say)* $\Rightarrow \quad r = b\sqrt{\sin 2\theta}$

Thus $\quad \mathbf{r^2 = b^2 \sin 2\theta}$, is the required O.T

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103. *Find the orthogonal trajectories of the family of curves* $\left(r + \frac{k^2}{r}\right) \cos\theta = a$, *a being the parameter*

>> We have $\left(r + \frac{k^2}{r}\right)\cos\theta = a$

$\Rightarrow \quad \log\left(r + \frac{k^2}{r}\right) + \log\cos\theta = \log a$

Differentiating *w.r.t* θ we have,

$$\frac{1}{\left(r + \frac{k^2}{r}\right)} \left(1 - \frac{k^2}{r^2}\right)\frac{dr}{d\theta} + \left(-\frac{\sin\theta}{\cos\theta}\right) = 0$$

ie., $\quad \frac{r}{r^2 + k^2} \cdot \frac{r^2 - k^2}{r^2} \frac{dr}{d\theta} = \tan\theta$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ we have,

$$\frac{r}{r^2 + k^2} \cdot \frac{r^2 - k^2}{r^2}\left(-r^2 \frac{d\theta}{dr}\right) = \tan\theta$$

ie., $\quad \frac{r(r^2 - k^2)}{r^2 + k^2}\left(-\frac{d\theta}{dr}\right) = \tan\theta$

or $\quad -\frac{d\theta}{\tan\theta} = \frac{r^2 + k^2}{r(r^2 - k^2)} dr$ by separating the variables

$$\Rightarrow \quad \int \frac{(r^2 + k^2)}{r(r^2 - k^2)} dr + \int \cot\theta \, d\theta = c \qquad \ldots (1)$$

But $\quad \frac{r^2 + k^2}{r(r - k)(r + k)} = \frac{-1}{r} + \frac{1}{r - k} + \frac{1}{r + k}$, by partial fractions.

$$\therefore \quad \int \frac{(r^2+k^2)\,dr}{r(r^2-k^2)} = -\log r + \log(r-k) + \log(r+k)$$

ie., $$\int \frac{(r^2+k^2)\,dr}{r(r^2-k^2)} = \log\left(\frac{r^2-k^2}{r}\right) \qquad \ldots (2)$$

Using (2) in (1) we have,

$$\log\left(\frac{r^2-k^2}{r}\right) + \log\sin\theta = c \quad \text{or} \quad \log\left(\frac{r^2-k^2}{r}\cdot\sin\theta\right) = \log b \ \ (say)$$

Thus $\left(r-\dfrac{k^2}{r}\right)\sin\theta = b$, is the required O.T.

104. *Using the concept of orthogonal trajectories show that the family of curves $r = a(\sin\theta+\cos\theta)$ and $r = b(\sin\theta-\cos\theta)$ intersect each other orthogonally.*

>> We shall show that the O.T of the first family $r = a(\sin\theta+\cos\theta)$ is the other family $r = b(\sin\theta-\cos\theta)$.

Consider $r = a(\sin\theta+\cos\theta)$

$$\Rightarrow \quad \log r = \log a + \log(\sin\theta+\cos\theta)$$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} \quad \text{or} \quad -r\frac{d\theta}{dr} = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$$

$$\therefore \quad \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\,d\theta = \frac{dr}{-r} \quad \text{by separating the variables.}$$

$$\Rightarrow \quad \int\frac{dr}{r} + \int\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\,d\theta = c$$

ie., $$\log r - \log(\cos\theta-\sin\theta) = c$$

ie., $$\log\left[\frac{r}{\cos\theta-\sin\theta}\right] = \log k \quad (say)$$

$$\Rightarrow \quad r = k(\cos\theta-\sin\theta) \quad \text{or} \quad r = -k(\sin\theta-\cos\theta)$$

Denoting $-k = b$ we have $r = b(\sin\theta-\cos\theta)$ **as required.**

105. [illegible]
$r(1+\cos\theta) = b$, where a, b are constants

>> Consider $r(1-\cos\theta) = a$

$\Rightarrow$ $\log r + \log(1-\cos\theta) = \log a$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ and simplifying the second term we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

ie., $r\dfrac{d\theta}{dr} = \cot(\theta/2)$ or $\tan(\theta/2)\,d\theta = \dfrac{dr}{r}$ by separating the variables

$\Rightarrow$ $\displaystyle\int\frac{dr}{r} - \int\tan(\theta/2)\,d\theta = c$

ie., $\log r - 2\log\sec(\theta/2) = c$

ie., $\log\left[\dfrac{r}{\sec^2(\theta/2)}\right] = \log k$ *(say)*

$\Rightarrow$ $r\cos^2(\theta/2) = k$ or $r\cdot\dfrac{1}{2}(1+\cos\theta) = k$

ie., $r(1+\cos\theta) = 2k$. and let $2k = b$.

Thus $r(1+\cos\theta) = b$, **is the required O.T.**

106. *Test for self orthogonality $r^n = a\sin n\theta$*

>> Consider $r^n = a\sin n\theta$

$\Rightarrow$ $n\log r = \log a + \log(\sin n\theta)$.

Differentiating *w.r.t* θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{n\cos n\theta}{\sin n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we have

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot n\theta \quad \text{or} \quad -r\frac{d\theta}{dr} = \cot n\theta$$

$\therefore \quad \tan n\theta \, d\theta = \dfrac{dr}{-r}$ by separating the variables.

$\Rightarrow \quad \displaystyle\int \frac{dr}{r} + \int \tan n\theta \, d\theta = c$

ie., $\quad \log r + \dfrac{\log(\sec n\theta)}{n} = c \quad$ or $\quad n \log r + \log(\sec n\theta) = nc$

ie., $\quad \log(r^n \sec n\theta) = \log b \quad (say)$

$\therefore$ **$r^n = b \cos n\theta$ is the required O.T and we conclude that the given family is not self orthogonal.**

107. [illegible]

$\dfrac{dv}{dr} = \dfrac{-du}{d\theta}$ *verify that the family of curves $u(r, \theta) = a$ and $v(r, \theta) = b$ are orthogonal trajectories of each other*

\>> With the background of polar curves we shall show that

$\tan\phi_1 \tan\phi_2 = -1$ where $\tan\phi = r\dfrac{d\theta}{dr}$ in general, ϕ being the angle between the radius vector and the tangent.

Consider $u(r, \theta) = a$ and differentiate w.r.t θ treating r as a function of θ

ie., $\quad u_r \dfrac{dr}{d\theta} + u_\theta = 0 \quad \therefore \quad \dfrac{dr}{d\theta} = \dfrac{-u_\theta}{u_r}$

We have $\tan\phi_1 = r\dfrac{d\theta}{dr}$ and hence $\tan\phi_1 = \dfrac{-r u_r}{u_\theta}$

Similarly for $v(r, \theta) = b$, we have $\tan\phi_2 = \dfrac{-r v_r}{v_\theta}$

$\therefore \quad \tan\phi_1 \cdot \tan\phi_2 = \dfrac{(r u_r)(r v_r)}{u_\theta \cdot v_\theta}$

But $\quad r u_r = v_\theta$ and $r v_r = -u_\theta$ by data.

Hence $\tan\phi_1 \cdot \tan\phi_2 = \dfrac{(v_\theta)\cdot(-u_\theta)}{u_\theta \cdot v_\theta} = -1$

Hence the family of curves $u(r, \theta) = a$ and $v(r, \theta) = b$ are orthogonal trajectories of each other.

EXERCISES

Find the orthogonal trajectories of the following family of curves [1 to 4]

1. $y = ax^2$ 2. $xy = a^2$ 3. $y = a(x+1)$

4. $x^2 + y^2 + 2\lambda y + c = 0$, λ being the parameter.

5. Show that the O.T. of the family of circles having their centres on the y - axis and also passing through the origin is a family of circles passing through the origin having their centres on the x - axis.

6. Show that the family of parabolas $x^2 = 4a(y+a)$ is self orthogonal.

Find the orthogonal trajectories of the following family of curves. [7 to 15]

7. $r = a\sec^2(\theta/2)$ 8. $r^2 = a^2\sin 2\theta$ 9. $r^n = a^n \operatorname{cosec} n\theta$

10. $r = a(1+\cos\theta)$ 11. $r^n = a^n(1+\cos n\theta)$ 12. $r = a(1-\sin\theta)$

13. $r = a\cos^2\theta$ 14. $r = a^\theta$

15. $\left(r - \dfrac{k^2}{r}\right)\sin\theta = a$, where a is the parameter.

ANSWERS

1. $x^2 + 2y^2 = c$ 2. $x^2 - y^2 = c$

3. $x^2 + y^2 + 2x + b = 0$ 4. $x^2 + y^2 + 2\mu + c = 0$, μ is arbitrary

7. $r = b\operatorname{cosec}^2(\theta/2)$ 8. $r^2 = b\cos 2\theta$

9. $r^n = b\sec n\theta$ 10. $r = b(1-\cos\theta)$

11. $r^n = b(1-\cos n\theta)$ 12. $r = b(1+\sin\theta)$

13. $r^2 = b\sin\theta$ 14. $(\log r)^2 + \theta^2 = b$

15. $\left(r + \dfrac{k^2}{r}\right)\cos\theta = b$

6.5 Methods of solving the D.E. at a glance

General form of the D.E $M(x, y)\,dx + N(x, y)\,dy = 0$

Form of the D.E	Method of solving / solution
I Variables separable form (Recapitulation)	
1. $f(x)\,g(y)\,dx + F(x)\,G(y)\,dy = 0$	Divide by $g(y)\,F(x)$ and integrate.
2. $\frac{dy}{dx} = f(ax+by+c)$	Put $ax+by+c = t$
3. $\frac{dy}{dx} = \frac{(ax+by)+c}{k(ax+by)+c'}$	Put $ax+by = t$
II **Homogeneous form**	
1. $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree with or without the involvement of terms with (y/x)	Write the D.E in the form $\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$ and put $y = vx$
If homogeneous functions are involved with x/y	Write $\frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)}$ and put $x = vy$
2. $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$, $\frac{a}{a'} \neq \frac{b}{b'}$	Put $x = X+h$, $y = Y+k$. With proper choice of h and k the D.E reduces to a homogeneous D.E in X and Y. Put $Y = VX$ and solve
III **Exact form**	
1. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ must be satisfied.	$\int M\,dx + \int N(y)\,dy = c$ is the solution
2. When $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then	Multiply the D.E with I.F to make it exact
(a) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$	$e^{\int f(x)\,dx}$ is the I.F
$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$	$e^{-\int g(y)\,dy}$ is the I.F
(b) $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$	$\frac{1}{Mx - Ny}$ is the I.F
(c) M and N involving terms of the form $x^a y^b$	$x^a y^b$ is the I.F where a and b are found such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

3.	Identifying the standard exact differentials and putting the D.E in the form $c_1 d\,[f_1(x, y)] + c_2 d\,[f_2(x, y)] + \cdots = 0$	$c_1 f_1(x, y) + c_2 f_2(x, y) + \cdots = c$ is the solution on integration
IV	**Linear form**	
1.	$\dfrac{dy}{dx} + Py = Q$ where P and Q are functions of x	**Solution :** $y e^{\int P\,dx} = \int Q e^{\int P\,dx} dx + c$
2.	$\dfrac{dx}{dy} + Px = Q$ where P and Q are functions of y.	**Solution :** $x e^{\int P\,dy} = \int Q e^{\int P\,dy} dy + c$
3.	$f'(y)\dfrac{dy}{dx} + f(y)P = Q$ where $P = P(x)$ and $Q = Q(x)$	Put $f(y) = t$ and differentiate *w.r.t* x
4.	$f'(x)\dfrac{dx}{dy} + f(x)P = Q$ where $P = P(y)$ and $Q = Q(y)$	Put $f(x) = t$ and differentiate *w.r.t* y.
5.	$\dfrac{dy}{dx} + Py = Qy^n$ where $P = P(x)$ and $Q = Q(x)$	Divide by y^n and put $y^{1-n} = t$ and diff. *w.r.t* x.
6.	$\dfrac{dx}{dy} + Px = Qx^n$ where $P = P(y)$ and $Q = Q(y)$	Divide by x^n and put $x^{1-n} = t$ and differentiate *w.r.t* y

6.6 Type recognition - A retrospect

After having discussed several methods for solving a D.E an important aspect is *type recognition*. We have seen that a number of problems can be solved by more than one method. The reader should carefully take a note of several remarks made while discussing various types, **recognize the type and think of easier options if any before solving the problem in a particular method.**

A mixed set of problems are drawn from various examination papers and are just analysed for spotting the befitting method. It is left as an exercise for the reader to complete the problems that are not worked.

ANALYSIS OF PROBLEMS

Solve the following differential equations

1. $\dfrac{dy}{dx} - \dfrac{y}{x} = \sin\left(\dfrac{y}{x}\right)$

>> Observe the (y/x) terms. The D.E is homogeneous.

Put $y = vx$ and solve. *[Worked Problem - 8]*

2. $\dfrac{dy}{dx} = \dfrac{7x-3y-7}{7y-3x+3}$

>> Observe that there is no common factor in the x, y terms.

Hence it is reducible to the homogeneous form But before doing in th.. method check for exactness.

$$(7x-3y-7)\,dx-(7y-3x+7)\,dy = 0$$

$$\frac{\partial M}{\partial y} = -3, \quad \frac{\partial N}{\partial x} = 3 \quad \text{The equation is not exact.}$$

So it is inevitable to solve by using $x = X + h$, $y = Y + k$ [*Worked Problem* 20]

---

3. $(1+y^2)\,dx + (\tan^{-1} y - x)\,dy = 0$

Observe that M is a function of y only and N contains x

Hence the equation is linear in x Write it in respect of $\dfrac{dx}{dy}$ and solve

[*Similar to worked Problem* - 64]

---

4. $(1+e^{x/y})\,dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0$

>> Observe (x/y) factors and that M and N are homogeneous functions of degree 0 The equation must be written in respect of $\dfrac{dx}{dy}$

Solve by putting $x = vy$. Incidently this is also an exact equation

[*Worked Problems* - 15, 35 *by both methods*]

---

5. $(x+\tan y)\,dy = \sin 2y\,dx$

Similar explanation as in Example 3 The equation is linear in x

[*Worked Problem* - 68]

6. $(y^2+y+x)\,dy - y\,dx = 0$

>> Same explanation as of problem-3.

$$\frac{dx}{dy} = \frac{y^2+y+x}{y} \quad \text{or} \quad \frac{dx}{dy} - \frac{x}{y} = \frac{y^2+y}{y}$$

ie., $\dfrac{dx}{dy} - \dfrac{x}{y} = (y+1)$ is linear in x.

Aliter . The equation is neither separable nor homogeneous

Let us try for exactness.

$$M = -y \quad N = (y^2 + y + x)$$

$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1. \quad \text{The equation is not exact.}$$

Let us continue further.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2 \qquad \therefore \ \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2}{-y} = \frac{2}{y} = g(y)$$

$$\therefore \qquad e^{-\int g(y)\,dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = 1/y^2 \ \text{ is an I.F.}$$

Multiplying by $1/y^2$ we can solve as an exact equation.

---

7. $\left[\, x \tan(y/x) - y \sec^2(y/x) \,\right] dx + x \sec^2(y/x)\, dy = 0$

\>> Instantly one should recognize it to be a homogeneous equation as we have (y/x) terms homogeneous of degree 1. Write the equation in respect of $\frac{dy}{dx}$ and solve by putting $y = vx$. Worked Problem - 10]

---

8. $y\,dx - x\,dy = \sqrt{x^2 + y^2}\,dx$

\>> Each of the terms are homogeneous function of degree 1.

This can be solved by putting $y = vx$ [Similar Worked Problem - 6]

---

9. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

\>> This is looking like a linear equation in y. We should get rid off $\cos y$ in R.H.S.

Dividing by $\cos y$ we get $\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$

Put $\sec y = t$, the equation reduces to a linear equation in t

[Worked Problem - 73]

---

10. $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$

No common factor in x, y terms and hence it is reducible to the homogeneous form. But before doing so, try for exactness.

$$(2x - y + 1)\,dx - (x + 2y - 3)\,dy = 0$$

$$M = 2x - y + 1 \quad \text{and} \quad N = -x - 2y + 3$$

$\dfrac{\partial M}{\partial y} = 1$ and $\dfrac{\partial N}{\partial x} = 1$ The equation is exact and we can write the solution instantly.

$$\int M\,dx + \int N(y)\,dy = c \qquad ie., \; x^2 - xy + x - y^2 + 3y = c.$$

Remark *The thought exhibited in this Example and the gain in completing the problem easily be carefully noted.*

---

11. $(1+x^2)\dfrac{dy}{dx} + 2xy - 6x^2 = 0$

>> Clearly in the standard form of a linear equation in y.

Dividing by $(1+x^2)$ we have,

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{6x^2}{1+x^2}$$

This can be solved by assuming the solution for the linear equation

---

12. $\dfrac{dy}{dx} - y\tan x = y^2 \sec x$

>> Clearly in the form which can be reduced to a linear equation (*Bernoulli's form*). Divide by y^2 and put $1/y = t$.

---

13. $(2+2x^2\sqrt{y})\,y\,dx + (x^2\sqrt{y}+2)\,x\,dy = 0$

>> Separation of variables and homogeneous types can be ruled out at once So, try exact.

$$\frac{\partial M}{\partial y} = 2 + 2x^2 \cdot \frac{3}{2}y^{1/2} = 2 + 3x^2\sqrt{y}$$

$$\frac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2 \quad \text{The equation is exact.}$$

We can solve the exact equation easily. [*Worked Problem* - 36]

---

14 [illegible]

$$\frac{dx}{dy} = \frac{x+2y^3}{y} \quad \text{or} \quad \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

>> This is a linear equation in x in the standard form which can be solved by assuming the solution for the linear equation.

---

15. $x \sin(y/x)\, dy = [\, y \sin(y/x) - x\,]\, dx$

>> At once this can be recognized as a homogeneous equation

We can solve by putting $y = vx$

---

16. [illegible]

>> It appears like a linear equation in y. So we have to get rid off y in the R.H.S. Dividing by y we get,

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = \frac{1}{x^2}$$

This can be solved by putting $\log y = t$.

---

17. $\left[\, y^2 e^{xy^2} + 4x^3 \,\right] dx + \left[\, 2xy\, e^{xy^2} - 3y^2 \,\right] dy = 0$

>> By ruling out the first two methods instantly we have to try for exactness Infact this is an exact equation [Worked Problem - 34]

---

18. [illegible]

>> We have to write the equation in the form

$$\frac{dx}{dy} = x^2 y^3 + xy \quad \text{or} \quad \frac{dx}{dy} - xy = x^2 y^3$$

We have to divide by x^2 and later put $1/x = t$. [Worked Problem - 80]

---

19. $x\dfrac{dy}{dx} + \dfrac{y}{x} = y$

>> This appears like a linear equation but we cannot put it in the proper form either dividing by x or by y. So let us simplify the problem.

$$x\frac{dy}{dx} = y - \frac{y^2}{x} \quad \text{or} \quad \frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

This is a homogeneous equation and can be solved by putting $y = vx$

[Worked Problem - 7]

---

20. $\dfrac{dy}{dx} = \dfrac{3y^2 + x^2}{2xy}$

>> At once we can recognize it to be a homogeneous equation and it can be solved by putting $y = vx$.

---

21. $(x^2 + y^2 - a^2)\, x\, dx + (x^2 - y^2 - b^2)\, y\, dy = 0$

>> We can easily rule out the first two methods and hence try for exactness. Infact this is an exact equation.

---

22. $(y^3 - 3x^2 y)\, dx - (x^3 - 3xy^2)\, dy = 0$

>> It can be easily recognized as a homogeneous equation. But we can attempt for checking the exactness. Infact this is an exact equation also.
We can easily write the solution of the exact equation.

[Worked Problem 3 & 26 in both the methods]

---

23. $x\dfrac{dy}{dx} + y = y^2 \log x$

>> It is appearing in the form of a linear equation and we have to divide by y^2 as well as by x.

We divide by $x\,y^2$ to obtain the equation in the form

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x}$$

Put $1/y = t$ to obtain a linear equation in t

---

24. $(y\cos x + \sin y + y)\, dx + (\sin x + x\cos y + x)\, dy = 0$

>> At once we can venture to try for exactness. Infact the equation is exact

[Worked Problem - 31]

---

25. $(x^2 y - 2xy^2)\, dx - (x^3 - 3x^2 y)\, dy = 0$

>> It can be recognized as a homogeneous equation at once. But it is not an exact equation. So we have to solve by putting $y = vx$.

---

26. $(2x - 10y^3)\, dy + y\, dx = 0$

>> This is linear in x and we put it in the form

$$\frac{dx}{dy} = \frac{10y^3 - 2x}{y} \quad \text{or} \quad \frac{dx}{dy} + \frac{2x}{y} = 10y^2$$

We can solve by using the associated standard solution

27. $(y - x + 1)\, dx - (y + x + 5)\, dy = 0$

>> Observe that there is no common factor in the x, y terms. So we can solve by reducing it to the homogeneous form by using the substitution $x = X + h$ and $y = Y + k$. This is inevitable as the equation is not exact.

28. $y \sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0$

At once we can rule out the first two methods and try for exactness. Infact the equation is exact.

29. $xy^2\, dy - (x^3 + y^3)\, dx = 0$

It can be easily recognized as a homogeneous equation and is not an exact one. Hence we have to solve by putting $y = vx$.

30. $x\, dy - y\, dx + \log y\, dy = 0$

We need to simplify the problem first i.e., $(x + \log y)\, dy = y\, dx$

$$\frac{dx}{dy} = \frac{x + \log y}{y} \quad \text{or} \quad \frac{dx}{dy} - \frac{x}{y} = \frac{\log y}{y}$$

This is a linear equation in x which can be solved by using the solution for the linear equation.

Unit - VII

LINEAR ALGEBRA - 1

7.1 Introduction

Linear Algebra broadly deals with theoretical and practical applications of linear transformations, linear system of equations etc. ***Matrix Theory*** is an important topic in linear algebra and the reader is acquainted with this to a certain extent. In this unit we discuss matrix oriented topics such as rank of a matrix, elementary transformation, consistency of a system of linear algebraic equations using the concept of rank of a matrix, solutions of a linear algebraic system of equations by some special methods.

7.2 Recapitulation of the basic matrix theory

A set of mn elements written in an array of m rows and n columns embedded in brackets [ ] or ( ) is called a ***matrix*** of order $m \times n$ (m by n). The following array is a typical $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots a_{mn} \end{bmatrix} = [a_{ij}]$$

Order of $A = O(A) = m \times n$

If $m = n$ the matrix is called a ***square matrix*** of order n

The matrix of order $1 \times n$ is called a ***row matrix*** and a matrix of order $n \times 1$ is called a ***column matrix***

Examples : $[1 \;\; 2 \;\; 3]$ . . . Row matrix (order 1×3)

$$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \quad \dots \text{ column matrix (order } 3 \times 1\text{)}$$

Consider a square matrix A of order n as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \ldots & & & & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}$$

In A the elements $a_{11}, a_{22}, \ldots a_{nn}$ constitute *principal diagonal* of the square matrix A.

The sum of all these elements is called the *trace* of A. A square matrix having all the elements below the principal diagonal zero is called an ***upper triangular matrix*** and a square matrix having all the elements above the principal diagonal zero is called a ***lower triangular matrix***

Examples : $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 8 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 6 & 7 & 8 \end{bmatrix}$ are respectively upper and lower triangular matrices

A square matrix is said to be a ***diagonal matrix*** if every element other than the principal diagonal elements are zero.

Examples : $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

If every element of a diagonal matrix is the same then it is called a ***scalar matrix.***

Examples : $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

A scalar matrix in which each diagonal element is equal to unity, that is 1 is called a ***Unit matrix*** or ***Identity matrix*** usually denoted by I.

Examples : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A matrix having all its elements equal to zero is called a ***null matrix.***

Examples : $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The matrix obtained by interchanging their rows and columns is called as the ***transpose*** of the given matrix usually denoted by A' where A is the given matrix

Obviously $(A')' = A$.

Examples : If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 4 \end{bmatrix}$

If $A = [\,2 \ \ -1 \ \ 4\,]$ then $A' = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

A square matrix A is said to be *symmetric* if $A = A'$ and *skew symmetric* if $A = -A'$.

Examples : $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 2 \\ 4 & 5 & 0 \\ 2 & 0 & 7 \end{bmatrix}$ are symmetric matrices.

(*observe that row elements and column elements are the same*)

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}$ are skew symmetric matrices.

(*observe that the principal diagonal elements are zero and column elements are negative of the row elements.*)

Algebra of matrices

If A is any matrix and k be any scalar the matrix obtained by multiplying every element of A by k is called the scalar multiple of the matrix A denoted by kA.

Examples :

If $A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 6 \end{bmatrix}$ then $3A = \begin{bmatrix} 6 & 9 & 15 \\ -3 & 12 & 18 \end{bmatrix}$, $\frac{1}{2} A = \begin{bmatrix} 1 & 3/2 & 5/2 \\ -1/2 & 2 & 3 \end{bmatrix}$

Sum and Difference of two matrices

If A and B are two matrices of the same order, their sum $A + B$, difference $A - B$ is the sum/difference of the corresponding elements.

Examples

If $A = \begin{bmatrix} -1 & 4 \\ 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 5 & -4 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 5 & 7 \\ 10 & 0 \end{bmatrix},\quad A - B = \begin{bmatrix} -7 & 1 \\ 0 & 8 \end{bmatrix}$$

$$2A + 3B = \begin{bmatrix} 16 & 17 \\ 25 & -4 \end{bmatrix},\quad A + \frac{1}{2} B = \begin{bmatrix} 2 & 11/2 \\ 15/2 & 2 \end{bmatrix}$$

Product of two matrices

If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$ then the product AB exists and will be a matrix of order $m \times p$.

Row elements of A are multiplied with the corresponding column elements of B and are added.

Examples :

1. Consider $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 6 \end{bmatrix}$

Order of A, $O(A) = 2 \times 3$, $O(B) = 3 \times 2$ $\therefore$ $O(AB) = 2 \times 2$

The product AB is computed as follows.

$$AB = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3\times1 + 2\times0 + 1\times3, & 3\times2 + 2\times4 + 1\times6 \\ 2\times1 + 4\times0 + 2\times3, & 2\times2 + 4\times4 + 2\times 6 \end{bmatrix} = \begin{bmatrix} 6 & 20 \\ 8 & 32 \end{bmatrix}$$

Also let us examine the possibility of the computation of the matrix product BA

$O(B) = 3 \times 2$, $O(A) = 2 \times 3$ $\therefore$ $O(BA) = 3 \times 3$

The product BA is computed as follows.

$$BA = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1\times3+2\times2, & 1\times2+2\times4, & 1\times1+2\times2 \\ 0\times3+4\times2, & 0\times2+4\times4, & 0\times1+4\times2 \\ 3\times3+6\times2, & 3\times2+6\times4, & 3\times1+6\times2 \end{bmatrix}$$

i.e., $$BA = \begin{bmatrix} 7 & 10 & 5 \\ 8 & 16 & 8 \\ 21 & 30 & 15 \end{bmatrix}$$

Remark *We will come across with product of two square matrices of the same order Obviously the resulting product matrix will also be a square matrix of the same order*

Note : Properties of matrix multiplication.

1. The matrix product AB may exist but BA may not exist Even if both exist, $AB \neq BA$ Matrix multiplication in general is not commutative
2. If $O(A) = m \times n$, $O(B) = n \times p$, $O(C) = p \times q$ then $A(BC) = (AB)C$ which implies that the matrix multiplication is associative.
3. If $O(A) = m \times n$ and $O(B) = n \times p = O(C)$ then $A(B+C) = AB + AC$, which implies that matrix multiplication is distributive
4. $(AB)' = B'A'$
5. If A is any square matrix, the matrix product $A \cdot A$ is denoted by A^2. Further $A \cdot A^2 = A^3$ and so-on.
   In general if p and q are positive integers,
   $A^p \cdot A^q = A^{p+q}$ and $(A^p)^q = A^{pq}$.

Orthogonal Matrix

A square matrix A is said to be ***orthogonal*** if $AA' \quad I = A'A$

Singular matrix and Nonsingular matrix

A square matrix A is said to be ***singular*** if its determinant is zero and A is said to be ***nonsingular*** if its determinant is not equal to zero.

That is, $|A| = 0 \Rightarrow A$ is singular and

$|A| \neq 0 \Rightarrow A$ is nonsingular.

Inverse of a square matrix

If A and B are two square matrices of the same order such that

$$AB = BA = I$$

then B is called the inverse of A denoted by A^{-1}.

The necessary and the suffecient condition for a square matrix A to possess inverse is that $|A| \neq 0$ That is to say that the inverse of a square matrix exists if and only if the matrix A is nonsingular

Note : 1. *The inverse of a square matrix is unique.*

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. *We have already said that* $AA' = I = A'A \Rightarrow A$ *is orthogonal.*

Comparing this with the definition of the inverse of a square matrix we can also conclude that

$$A' = A^{-1} \Rightarrow A \text{ is orthogonal.}$$

Adjoint of a square matrix

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the adjoint of A denoted by Adj A is given by

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Here A_{ij} represents the cofactor of a_{ij} in $|A|$. That is, $A_{ij} - (-1)^{i+j}$ multiplied by the value of the determinant obtained by deleting the i^{th} row and j^{th} column. However in the case of a second order square matrix, the cofactors will be a single element.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then Adj $A = \begin{bmatrix} +a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Now, for the third order square matrix A we have,

$$\text{Adj } A = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

Remark : *It is preferable to write the cofactors of elements column wise premultiplied by the signs* $+, -, +, -, +, -, +$ ... *and enter row wise in the matrix, resulting in the adjoint of the given matrix.*

Further the adjoint of a square matrix helps in finding the inverse of a nonsingular matrix A by the following established result.

$$A^{-1} = \frac{1}{|A|} \text{ Adj } A$$

Illustrative Examples to compute the inverse of a given square matrix.

1. $A = \begin{bmatrix} 1 & -2 \\ -3 & 8 \end{bmatrix}$

>> $|A| = \begin{vmatrix} 1 & -2 \\ -3 & 8 \end{vmatrix} = 8 - (+6) = 2$

$$\text{Adj } A = \begin{bmatrix} +8, & -(-2) \\ -(-3), & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix}$$

2. $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

>> $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = 1\,(-28+30) - 0 - 1\,(-18-0) = 20$

$$\text{Adj } A = \begin{bmatrix} +(-28+30), & -(0-6), & +(0+4) \\ -(-21-0), & +(-7-0), & -(5+3) \\ +(-18-0), & -(-6-0), & +(4-0) \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

7.3 Elementary transformations (operations) associated with a matrix

The following are the elementary ***row transformations*** of a matrix. The transformations can also be applied for columns.

1. Interchange of any two rows.
2. Multiplication of any row by a non zero constant.
3. Addition to any row a constant multiple of any other row.

Elementary transformations along with the notations we use is illustrated in the following table by considering the matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Example

	Elementary row transformation	Notation	Resultant of matrix A
1.	Interchange of first and second row	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
2.	Multiplication of third row by a constant k	$k R_3$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kc_1 & kc_2 & kc_3 \end{bmatrix}$
3.	Addition to second row k times the first row	$R_2 \to k R_1 + R_2$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ (ka_1+b_1) & (ka_2+b_2) & (ka_3+b_3) \\ c_1 & c_2 & c_3 \end{bmatrix}$

Equivalent matrices Two matrices A and B of the same order are said to be ***equivalent*** if one matrix can be obtained from the other by a finite number of successive elementary row (column) transformations. We use the notation $A \sim B$.

7.4 Echelon form and Normal form of a matrix

A non zero matrix A is said to be in ***row echelon form*** if the following conditions prevail

(a) All the zero rows are below non zero rows.

(b) The first non zero entry in any non zero row is 1.

(*Obviously in view of the condition (a) the entries below 1 in the same column are zero.*)

Example :

1. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Normal form (Canonical form)

The given matrix A is reduced to an echelon form first by applying a series of elementary row transformations.

Later column transformations are performed to reduce the matrix to one of the following four forms, called the **Normal form** of A.

(i) I_r (ii) $[I_r, \ 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

where I_r is the identity matrix of order r.

Observe the following corresponding examples of the normal form of a given matrix

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

These are equivalent to the following forms respectively.

(i) I_3 (ii) $[I_3, \ 0]$ (iii) $\begin{bmatrix} I_2 \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

Working procedure for problems

- In order to reduce the given matrix to a row echelon form we must prefer to have the leading entry (*first entry in the first row*) non zero, much prefarably 1
- In the case when this entry is zero we can interchange with any suitable row to meet the requirement.
- We then focus on the leading non zero entry (*starting from the first row*) to make all the elements in that column zero. However the transformation has to be performed for the entire row.

- Row echelon form will be achieved first and we have to continue further to arrive at the normal form
- We need to perform column transformations in the same way to achieve the normal form.

Note : *It is advisable to avoid fractions as far as possible during the process of elementary transformations.*

WORKED PROBLEMS

1. *Reduce the following matrix to the echelon form*

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & 8 \end{bmatrix}$$

>> Let $$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & 8 \end{bmatrix}$$

$$R_2 \to 2R_1 + R_2, \quad R_3 \to -R_1 + R_3$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & 8 \end{bmatrix}$$

$$R_3 \to 1/4 \cdot R_2 + R_3$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9/4 & 8 \end{bmatrix}$$

$$1/8 \cdot R_2, \quad 4/9 \cdot R_3$$

$\therefore$ $$A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 1 & 32/9 \end{bmatrix}$$ **is the row echelon form of A**

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2. *Applying elementary transformations, reduce the following matrix to the normal form*

$$\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

>> Let $$A = \begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 7 & 12 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$R_2 \rightarrow -3R_1 + R_2,\ R_3 \rightarrow -3R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(*We have to perform column transformations to reduce to the normal form*)

$C_2 \rightarrow -C_1 + C_2,\ C_3 \rightarrow -2C_1 + C_3,\ C_4 \rightarrow -3C_1 + C_4,\ C_5 \rightarrow -5C_1 + C_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow -C_2 + C_3,\ C_4 \rightarrow -2C_2 + C_4,\ C_5 \rightarrow -3C_2 + C_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$(-1)\ C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ **is the normal form of** A**.**

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3. *Find the normal form of the following matrix*

$$\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

>> Let $$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 0 & 2 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_2 \rightarrow -2R_1 + R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_3 \rightarrow 1/2 \cdot R_2 + R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $C_2 \rightarrow -1/2 \cdot C_1 + C_2$, $C_3 \rightarrow -3/2 \cdot C_1 + C_3$, $C_4 \rightarrow -1/2 \cdot C_1 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow -2C_2 + C_3$, $C_4 \rightarrow 2C_2 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/2 \quad C_1$, $-1/2 \cdot C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ **is the normal form of A.**

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4. *By performing elementary row and column transformations, reduce the following matrix to the normal form.*

$$\begin{matrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{matrix}$$

>> Let
$$A = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 1 & 4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$$R_2 \to -2R_1 + R_2, \ R_4 \to -4R_1 + R_4$$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & -2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

$$R_4 \to -R_2 + R_4$$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & -2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 1 & 9 & -4 \end{bmatrix}$$

$$R_4 \to -R_3 + R_4$$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & -2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $C_2 \to 2C_1 + C_2, \ C_3 \to -C_1 + C_3, \ C_4 \to 4C_1 + C_4, \ C_5 \to 2C_1 + C_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to C_2 + C_3, \ C_4 \to -3C_2 + C_3, \ C_5 \to -C_2 + C_5$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to -9\,C_3 + C_4, \;\; C_5 \to 4\,C_3 + C_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ **is the normal form of A.**

5 *Reduce the following matrix to its normal form*

>> Let

$$A = \begin{bmatrix} 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 9 & 3 & 1 & 0 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & -6 \\ 9 & 3 & 1 & 0 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

$R_2 \to -3\,R_1 + R_2, \;\; R_3 \to -9\,R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & -6 & -8 & -9 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

$R_3 \to -2\,R_2 + R_3, \;\; R_4 \to -2\,R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 5 & 27 \end{bmatrix}$$

$R_4 \to 5/4 \cdot R_3 + R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

Now, $C_2 \to -C_1 + C_2,\ C_3 \to -C_1 + C_3,\ C_4 \to -C_1 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

$$C_3 \to -2/3 \cdot C_2 + C_3,\ C_4 \to -3\,C_2 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

$$C_4 \to 9/4 \cdot C_3 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

Finally, $-1/3 \cdot C_2,\ -1/4 \cdot C_3,\ 4/153 \cdot C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4 \text{ **is the normal form of** } A.$$

7.5 Rank of a matrix

The *rank* of a matrix A in **echelon form** is equal to the ***number of non zero rows.*** It is denoted by $\rho(A)$.

Any matrix A of order $m \times n$ can be reduced to one of the normal forms .

(i) I_r (ii) $[\,I_r,\ 0\,]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

It is evident that the matrix in these normal forms will have r non zero rows.

Hence $\rho(A) = r$

If A is a $m \times n$ matrix of rank r, then there exists nonsingular matrices P and Q such that the matrix PAQ is in the normal form.

Further we can say that if r is the rank of a matrix A of order $m \times n$ $(r \le m)$, r number of rows of the matrix are linearly independent.

Observe the following examples.

1. $$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Performing, $R_2 \to -3R_1 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 1$$

This means that *one row* of the matrix A *is linearly independent*. It can be easily seen that $R_2 = 3R_1$ or $R_1 = 1/3\, R_2$

2. $$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 1 & 5 & 7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ yields,

$$A \sim \begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

$R_2 \to 2R_1 + R_2$, $R_3 \to R_1 + R_3$ gives us

$$A \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 7 & 10 \\ 0 & 7 & 10 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$A \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 7 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally $-R_1$ and $1/7 \cdot R_2$ yields

$$A \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 10/7 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 2$$

This means that *two rows* of the matrix A *are linearly independent*. It can be easily seen that $R_1 + R_2 = R_3$ or $R_2 = R_3 - R_1$ or $R_1 = R_3 - R_2$

Any one of the row of the matrix A is expressible in terms of the other two rows.

The application of the rank of a matrix is discussed in the next article

Remark : *We can write down the rank of the matrices in problems 1 to 5 discussed earlier*

Problem No.	1	2	3	4	5
$\rho(A)$	3	2	2	3	4

WORKED PROBLEMS

Find the rank of the following matrices by applying elementary row transformations [6 to 9]

6. $$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

>> $R_2 \to -2R_1 + R_2 \quad R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_2 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(-1)\, R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having *two* non zero rows

Hence the rank of A is 2. Thus $\rho(A) = 2$

7. $$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_3 \to -2R_1 + R_3$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/2 \cdot R_1,\ 1/2\ R_2$

$$A \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having *two* non zero rows.

Thus $\rho(A) = 2$

---

8. $A = \begin{bmatrix} \text{[illegible]} \\ \text{[illegible]} \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -9 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & -5 & -9 & 1 \end{bmatrix}$$

$R_3 \to 2R_2 + R_3,\ R_4 \to 5R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$ and $-1/4 \cdot R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having *three* non zero rows.

Thus $\rho(A) = 3$

>> *[Firstly, we prefer to interchange the first and second rows as it would be convenient to make the leading entry in the other rows zero]*

$R_2 \leftrightarrow R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2, \quad R_3 \to -R_1 + R_3, \quad R_4 \to -R_1 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_2 + R_3, \quad R_4 \to R_2 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

$R_4 \to -2R_3 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$1/2 \cdot R_1, \quad 1/2 \cdot R_2, \quad -1/3 \ R_3$

$$A \sim \begin{bmatrix} 1 & 1/2 & 3/2 & 2 \\ 0 & 1 & 2 & 7/2 \\ 0 & 0 & 1 & 4/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All the *four* rows are non zero in the row echelon form A, **Thus** $\rho(A) = 4$

[illegible] and [illegible] them to the normal form [illegible] { 10 to 13]

[illegible]

>> $R_1 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 0 & 2 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_3 \to 1/2 \ R_2 + R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to -1/2 \cdot C_1 + C_2, \quad C_3 \to -3/2 \ C_1 + C_3, \quad C_4 \to -1/2 \cdot C_1 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to -2 \cdot C_2 + C_3, \quad C_4 \to -2 \cdot C_2 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/2 \cdot C_1, \ -1/2 \cdot C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ **Thus $\rho(A) = 2$**

\>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to 2R_1 + R_2, \quad R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$$

$R_3 \to 2R_2 + R_3, \quad R_4 \to -3R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_4 \to R_3 + R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \rightarrow -2C_1 + C_2,\ C_3 \rightarrow -3C_1 + C_3,\ C_4 \rightarrow C_1 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow -C_2 + C_3,\ C_4 \rightarrow -C_2 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$ and $1/2 \cdot C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ Thus $\rho(A) = 3$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow -3R_1 + R_3,\ R_4 \rightarrow -R_1 + R_4$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3, \; R_4 \to -R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to -C_1 + C_3, \; C_4 \to -C_1 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 3C_2 + C_3, \; C_4 \to C_2 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ Thus $\rho(A) = 2$

---

13. $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

[We can perform elementary transformations like the earlier problems. However if one can observe that the fourth row is the sum of the first three rows, we can as well proceed by subtracting the fourth row from the sum of first three rows]

$$R_4 \to -(R_1+R_2+R_3)+R_4$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to -2R_1+R_2, \ \ R_3 \to -3R_1+R_3$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[*Preferring to avoid fraction let us perform* $-4R_2+5R_3$ *to make the leading entry in third row zero*]

$$R_3 \to -4R_2+5R_3$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[*It is advisible to multiply third row by* 1/11] $1/11 \ \ R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 \to C_1+C_2, \ \ C_3 \to 2C_1+C_3, \ \ C_4 \to 4C_1+C_4$$

Now, $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to -3/5 \cdot C_2+C_3, \ \ C_4 \to -7/5 \ \ C_2+C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4 \to 2/3\ C_3 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/5 \cdot C_2\,;\, 1/3 \cdot C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{Thus } \rho(A) = 3$$

---

14. *Find the values of k such that the matrix A may have rank equal to (a) 3 (b) 2*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix}$$

>> We have $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix}$

$$R_2 \to -R_1 + R_2\,,\ R_3 \to -R_1 + R_3$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & (k-1) \\ 0 & 3 & 9 & (k^2-1) \end{bmatrix}$$

$$R_3 \to -3R_2 + R_3$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & (k-1) \\ 0 & 0 & 0 & (k^2-3k+2) \end{bmatrix}$$

(a) Rank of A can be 3 if the equivalent form of A has 3 non zero rows

This is possible if $(k^2-3k+2) \neq 0$

ie., $\quad (k-1)(k-2) \neq 0$ or $k \neq 1$ and $k \neq 2$

Thus $\quad \boldsymbol{\rho(A) = 3}$ **if** $\boldsymbol{k \neq 1}$ **and** $\boldsymbol{k \neq 2}$

(b) Rank of A can be 2 if the equivalent form of A has 2 non zero rows

This is possible if $k^2 - 3k + 2 = 0$ or $(k-1)(k-2) = 0$

Thus $\rho(A) = 2$ if $k = 1$ or $k = 2$

---

15. [illegible] nonsingular matrices P and Q such that PAQ is in the normal form [illegible] the rank of A where

$$A = \begin{bmatrix} & & \\ 1 & -1 & -1 \\ & & \end{bmatrix}$$

Note : ***The following is the working procedure.***

- We assume $A = I\,A\,I$ where I is the identity matrix of the befitting order.
- We perform elementary transformations to reduce the matrix A to the normal form.
- The row transformations performed on A will also be performed on the first (prefactor of A) of the identity matrix whereas the column transformations performed on A will be performed on the second of the identity matrix (post factor of A).
- Thus we obtain the normal form of A equal to PAQ where P and Q are nonsingular matrices. We can instantly write $\rho(A)$ also.

Remark : **P and Q are not unique**

We now solve the given problem.

>> Let $A = I_3\, A\, I_3$ where I_3 represents identity matrix of order 3.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -R_1 + R_2,\ R_3 \to -3R_1 + R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $C_2 \to -C_1 + C_2,\ C_3 \to -C_1 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$-1/2 \cdot C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ,$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ **Also** $\rho(A) = 2$

---

16. [illegible] such that the matrix PAQ is in the normal form [illegible]

[illegible]

>> The order of A is 3×4 hence we need to take the prefactor of A as I_3 and post factor of A as I_4. That is,

$$A = I_3\ A\ I_4$$

ie.,
$$A = \begin{bmatrix} 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \\ 0 & 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 0 & 2 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/2 \cdot R_2 + R_3$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, $C_2 \to -1/2 \cdot C_1 + C_2$, $C_3 \to -3/2 \cdot C_1 + C_3$, $C_4 - \; -1/2 \cdot C_1 + C_4$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & -3/2 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -2\,C_2 + C_3$, $C_4 \to 2\,C_2 + C_4$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & -1/2 & -3/2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$1/2 \cdot C_1$, $-1/2 \cdot C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1/2 & 1/4 & -1/2 & -3/2 \\ 0 & -1/2 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$ **where** $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix}$ **and** $Q \begin{bmatrix} 1/2 & 1/4 & -1/2 & -3/2 \\ 0 & -1/2 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7.6 Consistency of a system of linear equations

A system of equations in which all the unknowns quantities appear in the first degree alone is called a *linear system of equations.*

A set of m linear equations in n unknown is as follows.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots \quad \dots \quad \dots \quad &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij}'s and b_i's are constants.

If $b_1, b_2, \dots b_m$ are all zero, the system is said to be *homogeneous*.

The set of values $x_1, x_2, \dots x_n$ which satisfy all the equations simultaneously is called a ***solution*** of the system of equations.

A system of linear equations is said to be ***consistent*** if it possess a solution. Otherwise the system is said to be ***inconsistent.***

The above system of equations can be written in the matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

That is, $AX = B$ where the matrix A is of order $m \times n$, X is of order $n \times 1$ and hence their product denoted by B is of order $m \times 1$. Obviously $AX = [\,0\,]$ is the matrix representation of the homogeneous set of equations where $[0]$ is a null matrix of order $m \times 1$.

$x_1 = x_2 = x_3 = \dots = x_n = 0$ is obviously a solution of the homogeneous system of equations and is called a ***trivial solution***. If at least one x_i ($i = 1, 2, \dots n$) is not equal to zero then it is called a ***non trivial solution***.

The system of equations $AX = B$ may or may not possess a solution. If the system possesses a solution it may or may not be the only solution. The concept of the rank of a matrix helps us to conclude (i) whether the system is consistent or not (ii) whether the system possess unique solution or many solutions.

Condition for consistency and types of solution

Consider a system of m equations in n unknowns represented in the matrix form $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdot & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the ***coefficient matrix.***

The matrix formed by appending to A an extra column consisting of the elements of B is called the ***augmented matrix*** denoted by $[A : B]$

i.e., $$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ & \cdot & & & \cdot & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

The system of equations represented by the matrix equation $AX = B$ ***is consistent if*** $\rho(A) = \rho[A:B]$

Suppose $\rho[A] = \rho[A:B] = r$ then the condition for various types of solution are as follows

1. ***Unique solution*** : $\rho[A] = \rho[A:B] = r = n$, n being the number of unknowns.

2. ***Infinite solutions*** : $\rho[A] = \rho[A:B] = r < n$

In this case $(n-r)$ unknowns can take arbitrary values. Obviously $\rho[A] \neq \rho[A:B]$ implies that the system is ***inconsistent*** (*does not possess a solution*)

Working procedure for problems

- We first form the augmented matrix $[A:B]$ and we can clearly identify the portion of the coefficient matrix A in it
- We reduce the matrix $[A:B]$ to an echelon form by elementary row transformations. This will enable us to immediately write down the ranks of A and also $[A:B]$ with the result we can decide the consistency aspect of the system of equations.
- The echelon form of $[A:B]$ is converted back to the equation form and the solution will emerge easily.

WORKED PROBLEMS

17. *Test for consistency and solve :*

$$x+y+z=6$$
$$x-y+2z=5$$
$$3x+y+z=8$$

>> $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & -1 & 2 & : & 5\\ 3 & 1 & 1 & : & 8\end{bmatrix}$ is the augmented matrix.

We now perform elementary row operations.

$$R_2 \to -R_1+R_2,\quad R_3 \to -3R_1+R_3$$

$$[A:B] \sim \begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & -2 & 1 & : & -1\\ 0 & -2 & -2 & : & -10\end{bmatrix}$$

$$R_3 \to -R_2+R_3$$

$$[A:B] \sim \begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & -2 & 1 & : & -1\\ 0 & 0 & 3 & : & -9\end{bmatrix}$$

Note : *We need not make the leading non zero entry in every row 1 as we can decide on the rank of the matrices A and [A : B] at this stage.*

Both A and $[A:B]$ matrices have all the three rows non zero

$\therefore$ $\rho[A]=3$ and $\rho[A:B]=3$ *ie.,* $r=3$

Also the number of independent variables $n=3$

Since $\rho[A]=\rho[A:B]=3$ (*ie.,* $r=n=3$) **the given system of equations is consistent and will have unique solution.**

Let us now convert the prevailing form of $[A:B]$ into a set of equations as follows.

$$x+y+z=6 \quad \ldots (i)$$

$$-2y+z=-1 \quad \ldots (ii)$$

$$-3z=-9 \quad \ldots (iii)$$

From (iii) $z=3$, substituting this value in (ii) we get $y=2$. Finally substituting these values in (i) we get $x=1$

Thus **$x=1, y=2, z=3$** is the unique solution

18. *Test for consistency and solve*

$$x + 2y + 3z = 14$$

$$4x + 5y + 7z = 35$$

$$3x + 3y + 4z = 21$$

>> Now $[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 4 & 5 & 7 & : & 35 \\ 3 & 3 & 4 & : & 21 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -4R_1 + R_2, \;\; R_3 \to -3R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & -3 & -5 & : & -21 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 2$, $\rho[A:B] = 2$ *i.e.,* $r = 2$. Also $n = 3$

Since $\rho[A] = \rho[A:B] = 2 < 3$ (*ie,* $r < n$) **the system is consistent and will have infinite solutions.** Here $(n - r) = 1$ and hence one of the variables can take arbitrary values.

We now have, $\quad x + 2y + 3z = 14 \qquad \ldots \text{(i)}$

$$-3y - 5z = -21 \qquad \ldots \text{(ii)}$$

Let $z = k$ be arbitrary.

$\therefore$ from (ii) $-3y - 5k = -21$ or $y = \dfrac{21 - 5k}{3} = 7 - \dfrac{5k}{3}$

Now from (i) $x + 2(7 - 5k/3) + 3k = 14 \quad \therefore \; x = k/3$

Thus $x = \dfrac{k}{3}$, $y = 7 - \dfrac{5k}{3}$, $z = k$ represent infinite solutions since k is arbitrary.

19. *Test for consistency and solve :*

$$x - 4y + 7z = 14$$

$$3x + 8y - 2z = 13$$

$$7x - 8y + 26z = 5$$

>> $[A : B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 3 & 8 & -2 & : & 13 \\ 7 & -8 & 26 & : & 5 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -3R_1 + R_2, \quad R_3 \to -7R_1 + R_3$

$$[A : B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 20 & -23 & : & -93 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A : B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 0 & 0 & : & -64 \end{bmatrix}$$

We have $\rho[A] = 2$ and $\rho[A : B] = 3$

Since $\rho[A] \neq \rho[A : B]$ the system is **inconsistent** (*does not possess any solution*)

Remark *By the conversion of the above form of the matrix to equations, the last row will result in an identity of the form $0 = -64$ which is absurd. Hence no solution for the system.*

---

20. *Test for consistency and solve :*

$$5x_1 + x_2 + 3x_3 = 20$$

$$2x_1 + 5x_2 + 2x_3 = 18$$

$$3x_1 + 2x_2 + x_3 = 14$$

>> $[A : B] = \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 2 & 5 & 2 & : & 18 \\ 3 & 2 & 1 & : & 14 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -2R_1 + 5R_2, \quad R_3 \to -3R_1 + 5R_3$

$$[A : B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 7 & -4 & : & 10 \end{bmatrix}$$

$R_3 \to -7R_2 + 23R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 0 & -120 & : & -120 \end{bmatrix}$$

We have $\rho[A] = 3$, $\rho[A:B] = 3$ *ie.,* $r = 3$. Also $n = 3$

Since $\rho[A] = \rho[A:B] = 3$ (*i.e.,* $r = n = 3$) **the system is consistent and will have unique solution.**

We now have, $5x_1 + x_2 + 3x_3 = 20$ ... (i)

$$23x_2 + 4x_3 = 50 \quad \ldots \text{(ii)}$$

$$-120x_3 = -120 \quad \ldots \text{(iii)}$$

$\therefore$ from (iii), $x_3 = 1$

From (ii) we get $x_2 = 2$ and from (i) we get $x_1 = 3$

Thus $\mathbf{x_1 = 3,\ x_2 = 2,\ x_3 = 1}$ is the unique solution.

---

21. *Test for consistency and solve*

$$x + 2y + 2z = 1$$

$$2x + y + z = 2$$

$$3x + 2y + 2z = 3$$

$$y + z = 0$$

>> $[A:B] = \begin{bmatrix} 1 & 2 & 2 & & 1 \\ 2 & 1 & 1 & : & 2 \\ 3 & 2 & 2 & : & 3 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -2R_1 + R_2$, $R_3 \to -3R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & -3 & -3 & : & 0 \\ 0 & -4 & -4 & & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & -4 & -4 & : & 0 \\ 0 & -3 & -3 & : & 0 \end{bmatrix}$$

$R_3 \to 4R_2 + R_3\,,\ \ R_4 \to 3R_2 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 2,\ \rho[A:B] = 2$ *i.e.,* $r = 2$. Also $n = 3$.

Since $\rho[A] = \rho[A:B] = 2 < 3$ (*i.e.,* $r < n$) **the system is consistent and will have infinite solutions.**

We now have, $\quad x + 2y + 2z = 1 \quad \ldots$ (i)

$y + z = 0 \quad \ldots$ (ii)

Let $z = k$ be arbitrary $\therefore\ y = -k$ from (ii).

Also from (i), $x + (-2k) + 2k = 1 \ \therefore\ x = 1$

Thus $x = 1\,,\ y = -k\,,\ z = k$ are the infinite number of solutions since k is arbitrary.

---

22. Show that the following system of equations does not possess any solution

$$5x + 3y + 7z = 5$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

>> $[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -3R_1 + 5R_2\,,\ \ R_3 \to -7R_1 + 5R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & -11 & 1 & : & -10 \end{bmatrix}$$

$R_3 \to R_2 + 11R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & 0 & 0 & : & -80 \end{bmatrix}$$

We have $\rho[A] = 2$, $\rho[A:B] = 3$

Since $\rho[A] \neq \rho[A:B]$ **the system is inconsistent.**

---

23. *Test for consistency and solve .*

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

>> $[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$ is the augmented matrix.

$$R_2 \to -3R_1 + 5R_2, \quad R_3 \to -7R_1 + 5R_3$$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 121 & -11 & : & 33 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$$R_2 \to -1/11 \cdot R_2$$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$$R_3 \to R_2 + R_3$$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 2$, $\rho[A:B] = 2$

Since $\rho[A] = \rho[A:B] = 2 < 3$ we conclude **that the system is consistent and will have infinite solutions.**

We now have, $5x + 3y + 7z = 4$ ... (i)

$$11y - z = 3 \quad \text{... (ii)}$$

Let $z = k$ be arbitrary and from (ii) $y = \frac{1}{11}(k+3)$

Also from (i), $5x + \frac{3}{11}(k+3) + 7k = 4 \quad \therefore \; x = \frac{1}{11}(7 - 16k)$

Thus $x = \frac{1}{11}(7 - 16k), \; y = \frac{1}{11}(k+3), \; z = k$ is the required solution

---

24. *Test for consistency and solve*

$$\begin{aligned} 4x - 5y + z &= -3 \\ 2x + 3y - z &= 3 \\ 3x - y + 2z &= 5 \\ x + 2y - 5z &= -9 \end{aligned}$$

\>> $[A : B] = \left[\begin{array}{ccc:c} 4 & -5 & 1 & -3 \\ 2 & 3 & -1 & 3 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & -5 & -9 \end{array}\right]$ is the augmented matrix

$R_1 \leftrightarrow R_4$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -5 & -9 \\ 2 & 3 & -1 & 3 \\ 3 & -1 & 2 & 5 \\ 4 & -5 & 1 & -3 \end{array}\right]$$

$R_2 \to -2R_1 + R_2, \; R_3 \to -3R_1 + R_3, \; R_4 \to -4R_1 + R_4$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -5 & -9 \\ 0 & -1 & 9 & 21 \\ 0 & -7 & 17 & 32 \\ 0 & -13 & 21 & 33 \end{array}\right]$$

$R_3 \to -7R_2 + R_3, \; R_4 \to -13R_2 + R_4$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -5 & -9 \\ 0 & -1 & 9 & 21 \\ 0 & 0 & -46 & -115 \\ 0 & 0 & -96 & -240 \end{array}\right]$$

$-1/23 \cdot R_3, \; -1/48 \cdot R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -5 & : & -9 \\ 0 & -1 & 9 & : & 21 \\ 0 & 0 & 2 & : & 5 \\ 0 & 0 & 2 & : & 5 \end{bmatrix}$$

$R_4 \to -R_3 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -5 & : & -9 \\ 0 & -1 & 9 & : & 21 \\ 0 & 0 & 2 & : & 5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 3$, $\rho[A:B] = 3$ *i.e.,* $r = 3$, also $n = 3$.

Since $\rho[A] = \rho[A \cdot B] = 3$ (*i.e.,* $r = n = 3$) **the system is consistent and will have unique solution.**

We now have

$$x + 2y - 5z = -9 \quad \ldots \text{(i)}$$

$$-y + 9z = 21 \quad \ldots \text{(ii)}$$

$$2z = 5 \quad \ldots \text{(iii)}$$

From (iii), $z = \frac{5}{2}$ $\therefore$ (ii) becomes $y = \frac{45}{2} - 21$ or $y = \frac{3}{2}$

Also from (i), $x + 3 - \frac{25}{2} = -9$ $\therefore$ $x = \frac{1}{2}$

Thus $\boldsymbol{x = \frac{1}{2},\ y = \frac{3}{2},\ z = \frac{5}{2}}$ is the unique solution.

---

25. *Investigate the values of* λ *and* μ *such that the system of equations*

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu, \text{ may have}$$

(*a*) *Unique solution* (*b*) *Infinite solution* (*c*) *No solution*

>> $[A \cdot B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$ is the augmented matrix

$R_2 \to -R_1 + R_2,\ R_3 \to -R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

(a) *Unique solution* . We must have $\rho[A] = \rho[A:B] = 3$, $\rho[A]$ will be 3 if $(\lambda-3) \neq 0$ since the other two entries in the last row of A are zero If $(\lambda-3) \neq 0$ or $\lambda \neq 3$ irrespective of the value of μ, $\rho[A:B]$ will also be 3.

$\therefore$ the system will have **unique solution if $\lambda \neq 3$**

(b) *Infinite solutions* : Here we have $n = 3$ and we need $\rho[A] = \rho[A:B] = r < 3$ We must have $r = 2$ since first row and second row are non zero.

$\therefore$ $\rho[A] = \rho[A:B] = 2$ only when the last row of $[A:B]$ is completely zero. This is possible if $\lambda-3 = 0$, $\mu-10 = 0$

$\therefore$ the system will have **infinite solution if $\lambda = 3$ and $\mu = 10$**

(c) *No solution* : We must have $\rho[A] \neq \rho[A:B]$. By case (a) $\rho[A] = 3$ if $\lambda \neq 3$ and hence if $\lambda = 3$ we obtain $\rho[A] = 2$.
If we impose $(\mu-10) \neq 0$ then $\rho[A:B]$ will be 3.

$\therefore$ the system has **no solution if $\lambda = 3$ and $\mu \neq 10$**

---

26. *Find for what values of k the system of equations*

$$x + y + z = 1$$

$$x + 2y + 4z = k$$

$$x + 4y + 10z = k^2,$$

possesses a solution. Solve completely in each case

>> $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & k \\ 1 & 4 & 10 & : & k^2 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -R_1 + R_2$, $R_3 \to -R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 3 & 9 & : & k^2-1 \end{bmatrix}$$

$$R_3 \to -3R_2 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 0 & 0 & : & k^2-3k+2 \end{bmatrix}$$

$\rho[A] = 2$ and for the system to be consistent we must have $\rho[A:B]$ also 2. This is possible if $k^2-3k+2 = 0 \quad \therefore \quad k = 1, \ k = 2.$

Hence we conclude that the system possesses a solution if $k = 1, 2$.

Since $\rho[A] = \rho[A:B] = 2 < 3$ for the cases $k = 1, 2$ the system will have infinite solution and the same are as follows.

Case - i: $k = 1$. The system of equations are,

$$x+y+z = 1 \qquad \ldots (i)$$

$$y+3z = 0 \qquad \ldots (ii)$$

Let $z = k_1$ be arbitrary $\therefore$ from (ii) $y = -3k_1$ and from (i) $x = 1+2k_1$

Case - ii: $k = 2$. The system of equations are

$$x+y+z = 1 \qquad \ldots (iii)$$

$$y+3z = 1 \qquad \ldots (iv)$$

Let $z = k_2$ be arbitrary $\therefore$ from (iv) $y = 1-3k_2$ and from (iii) $x = 2k_2$

Thus $x = 1+2k_1, \ y = -3k_1, \ z = k_1$ and $x = 2k_2, \ y = 1-3k_2, \ z = k_2$

give all the solution of the given system of equations.

---

27. *Show that the system of equations*

$$3x+4y+5z = a$$

$$4x+5y+6z = b$$

$$5x+6y+7z = c$$

possesses a solution if $a+c = 2b$. Solve when $(a, b, c) = (1, 2, 3)$

>> $[A:B] = \begin{bmatrix} 3 & 4 & 5 & : & a \\ 4 & 5 & 6 & : & b \\ 5 & 6 & 7 & : & c \end{bmatrix}$ is the augmented matrix.

Observing the elements in A, we perform the transformations : $R_3 \to -R_2 + R_3$, $R_2 \to -R_1 + R_2$ for convenience (*We can as well perform* $R_2 \to -4R_1 + 3R_3$ *and* $R_3 \to -5R_1 + 3R_3$)

$$[A:B] \sim \begin{bmatrix} 3 & 4 & 5 & : & a \\ 1 & 1 & 1 & : & b-a \\ 1 & 1 & 1 & : & c-b \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & b-a \\ 3 & 4 & 5 & : & a \\ 1 & 1 & 1 & : & c-b \end{bmatrix}$$

$R_2 \to -3R_1 + R_2, \; R_3 \to -R_1 + R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & b-a \\ 0 & 1 & 2 & : & 4a-3b \\ 0 & 0 & 0 & : & a-2b+c \end{bmatrix}$$

Here $\rho[A] = 2$ and the system possesses a solution if $\rho[A:B]$ is also equal to 2 which is possible only when $a - 2b + c = 0$ or $a + c = 2b$ as required

We shall solve the system when $a = 1, \; b = 2, \; c = 3$

For these values we have, $3x + 4y + 5z = 1$ ... (i)

$y + 2z = -2$ ... (ii)

Let $z = k$ be arbitrary. From (ii) $y = -(2 + 2k)$

From (i) : $3x - 8 - 8k + 5k = 1 \quad \therefore \; x = k + 3$

Thus $x = k+3, \; y = -2(1+k), \; z = k$ **is the required solution when** $(a, b, c) = (1, 2, 3)$

Solution of linear homogeneous equations

Let $AX = 0$ respresent the matrix representation of m linear homogeneous equations in n variables. The augmented matrix is given by

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & : 0 \end{bmatrix}$$

If $\rho[A] = \rho[A:B] = n$, the system is consistent and will have **trivial solution** $x_1 = 0 = x_2 = x_3 = \cdots = x_n$

If $\rho[A] = \rho[A:B] = r < n$ the system will have **non trivial, infinite number of solutions** as $(n-r)$ variables can be chosen arbitrarily.

WORKED PROBLEMS

28 *Find the value of λ for the following system of equations to have a non trivial solution. Also find the non trivial solution*

$$x + y + 3z = 0$$

$$4x + 3y + \lambda z = 0$$

$$2x + y + 2z = 0$$

>> $[A:B] = \begin{bmatrix} 1 & 1 & 3 & : & 0 \\ 4 & 3 & \lambda & : & 0 \\ 2 & 1 & 2 & : & 0 \end{bmatrix}$ is the augmented matrix.

$$R_2 \to -4R_1 + R_2,\ R_3 \to -2R_1 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 3 & : & 0 \\ 0 & -1 & (\lambda-12) & : & 0 \\ 0 & -1 & -4 & : & 0 \end{bmatrix}.$$

$$R_3 \to -R_2 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 3 & : & 0 \\ 0 & -1 & (\lambda-12) & : & 0 \\ 0 & 0 & (8-\lambda) & : & 0 \end{bmatrix}$$

Here $n = 3$ and we need to have $\rho[A] = \rho[A:B] = r < 3$

If $8-\lambda = 0$, then $\rho[A] = \rho[A:B] = 2 < 3$

Thus when $\lambda = 8$, the system will have non trivial solution.

Now, when $\lambda = 8$ we have the system of equations

$$x + y + 3z = 0 \qquad \text{(i)}$$

$$-y - 4z = 0 \qquad \text{(ii)}$$

Let $z = k$ be arbitrary $\therefore\ y = -4k$ from (ii).

Also from (i), $x - 4k + 3k = 0\ \therefore\ x = k$

Thus $x = k,\ y = -4k$ **and** $z = k$ **represents the non trivial solution** of the given system of equations when $\lambda = 8$

29. *Find the value of k such that the following system of equations possess a nontrivial solution. Also find the solution of the system*

$$4x_1 + 9x_2 + x_3 = 0$$

$$kx_1 + 3x_2 + kx_3 = 0$$

$$x_1 + 4x_2 + 2x_3 = 0$$

>> $[A : B] = \begin{bmatrix} 4 & 9 & 1 & : & 0 \\ k & 3 & k & : & 0 \\ 1 & 4 & 2 & : & 0 \end{bmatrix}$ is the augmented matrix.

$R_1 \leftrightarrow R_3$

$$[A : B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ k & 3 & k & : & 0 \\ 4 & 9 & 1 & : & 0 \end{bmatrix}$$

$R_2 \to -kR_1 + R_2,\ R_3 \to -4R_1 + R_3$

$$[A : B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ 0 & (3-4k) & -k & : & 0 \\ 0 & -7 & -7 & : & 0 \end{bmatrix}$$

$-1/7 \cdot R_3$ and then $R_2 \leftrightarrow R_3$

$$[A : B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & (3-4k) & -k & : & 0 \end{bmatrix}$$

$R_3 \to (4k-3)\ R_2 + R_3$

$$[A : B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & (3k-3) & : & 0 \end{bmatrix}$$

$\rho[A] = \rho[A : B] = 2 < 3$, only when $3k - 3 = 0$ or $k = 1$

Thus when $k = 1$, the system will have non trivial solution.

Now, when $k = 1$, we have the system of equations,

$$x + 4y + 2z = 0 \qquad \ldots \text{(i)}$$

$$y + z = 0 \qquad \ldots \text{(ii)}$$

Let $z = k$ be arbitrary. $\therefore\ y = -k$ from (ii).

Also from (i), $x - 4k + 2k = 0 \quad \therefore\ x = 2k$

Thus $x = 2k,\ y = -k$ and $z = k$, is the required non trivial solution

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$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 2 & 3 & 1 & : & 0 \\ 4 & 5 & 4 & : & 0 \\ 1 & 1 & -2 & : & 0 \end{bmatrix}$$ is the augmented matrix.

$R_2 \to -2R_1 + R_2,\ R_3 \to -4R_1 + R_3,\ R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & -1 & -5 & : & 0 \\ 0 & -3 & -8 & : & 0 \\ 0 & -1 & -5 & : & 0 \end{bmatrix}$$

$R_3 \to -3R_2 + R_3,\ R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & -1 & -5 & : & 0 \\ 0 & 0 & 7 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho[A] = \rho[A:B] = 3 =$ number of un knowns.

Hence the system is consistent and will have **trivial solution** :
$x = 0,\ y = 0,\ z = 0$

$$[A:B] = \begin{bmatrix} 1 & 1 & -1 & 1 & : & 0 \\ 1 & -1 & 2 & -1 & : & 0 \\ 3 & 1 & 0 & 1 & : & 0 \end{bmatrix}$$

$R_2 \to -R_1 + R_2,\ R_3 \to -3R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & -1 & 1 & : & 0 \\ 0 & -2 & 3 & -2 & : & 0 \\ 0 & -2 & 3 & -2 & : & 0 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & -1 & 1 & : & 0 \\ 1 & -2 & 3 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho[A] = \rho[A:B] = 2 < 4$, 4 being the number of unknowns.

The system does possess non trivial solution. $(4-2) = 2$ unknowns can be chosen arbitrarily.

We now have the system of equations,

$$x_1 + x_2 - x_3 + x_4 = 0 \qquad \ldots \text{(i)}$$

$$-2x_2 + 3x_3 - 2x_4 = 0 \qquad \ldots \text{(ii)}$$

Let $x_4 = k_1$ and $x_3 = k_2$ be arbitrary.

From (ii), $-2x_2 + 3k_2 - 2k_1 = 0 \;\therefore\; x_2 = \frac{1}{2}(3k_2 - 2k_1)$

From (i), $x_1 + \frac{1}{2}(3k_2 - 2k_1) - k_2 + k_1 = 0 \;\therefore\; x_1 = -\frac{1}{2} k_2$

Thus $x_1 = -\frac{1}{2} k_2,\; x_2 = \frac{3}{2} k_2 - k_1,\; x_3 = k_2,\; x_4 = k_1$ is the required non trivial solution.

>> $[A:B] = \begin{bmatrix} 1 & -2 & 1 & -1 & : & 0 \\ 1 & 1 & -2 & 3 & : & 0 \\ 4 & 1 & -5 & 8 & : & 0 \\ 5 & -7 & 2 & -1 & : & 0 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -R_1 + R_2,\; R_3 \to -4R_1 + R_3,\; R_4 \to -5R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 1 & -1 & : & 0 \\ 0 & 3 & -3 & 4 & : & 0 \\ 0 & 9 & -9 & 12 & : & 0 \\ 0 & 3 & -3 & 4 & : & 0 \end{bmatrix}$$

$$R_3 \to -3R_2 + R_3, \quad R_4 \to -R_2 + R_4$$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 1 & -1 & : & 0 \\ 0 & 3 & -3 & 4 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A:B] = 2$ and $\rho[A] = 2$ *i.e.*, $r = 2$. Also $n = 4$.

Since $\rho[A:B] = \rho[A] = 2 < 4$ the system is consistent and will have infinite solutions by choosing $(n-r)$ variables (*i.e.*, 2 variables) arbitrarily.

We now have

$$x_1 - 2x_2 + x_3 - x_4 = 0 \qquad \ldots \text{(i)}$$

$$3x_2 - 3x_3 + 4x_4 = 0 \qquad \ldots \text{(ii)}$$

Let us choose $x_4 = k_1$, $x_3 = k_2$ arbitrarily.

$\therefore$ from (ii), $3x_2 - 3k_2 + 4k_1 = 0$ or $x_2 = k_2 - \frac{4}{3}k_1$

Also from (i), $x_1 - 2k_2 + \frac{8}{3}k_1 + k_2 - k_1 = 0$ or $x_1 = k_2 - \frac{5}{3}k_1$

Thus $x_1 = k_2 - \frac{5}{3}k_1$, $x_2 = k_2 - \frac{4}{3}k_1$, $x_3 = k_2$, $x_4 = k_1$

give all the solutions of the given system of equations.

Find the rank of the following matrices by elementary row transformations.

1. $\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 2 & 5 & 2 \\ 2 & -1 & 1 & -3 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & 5 & -4 & 7 \\ -1 & -2 & -1 & 2 \\ 3 & 3 & -5 & 10 \end{bmatrix}$

6. $\begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & 1 & 1 & 4 \end{bmatrix}$

Find the rank of the following matrices by reducing it to the normal form

7. $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 1 & 3 & 3 & 11 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 2 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 4 & 6 & 4 & 6 \\ 7 & 8 & 5 & 10 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 6 \\ 3 & 5 & 6 & 10 \\ 4 & -1 & -3 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

Test for consistency and solve the following system of equations

11. $x+y+z = 9$
$2x+5y+7z = 52$
$2x+y-z = 0$

12. $x-y+2z = 3$
$x+2y+3z = 5$
$3x+4y-5z = -13$

13. $5x+y+3z = 20$
$2x+5y+2z = 18$
$3z+2y+z = 14$

14. $x+3y-2z = 0$
$2x-y+4z = 0$
$x-11y+14z = 0$

15. $4x-2y+6z = 8$
$x+y-3z = -1$
$15x-3y+9z = 21$

16. $x+y+z = 1$
$x+2y+3z = 4$
$x+3y+5z = 7$
$x+4y+7z = 10$

17. $2x+6y+11 = 0$
$6x+20y-6z+3 = 0$
$6y-18z+1 = 0$

18. $2x-3y+7z = 5$
$3x+y-3z = 13$
$2x+19y-47z = 32$

19. Find the values of λ and μ such that the following system of equations, $2x+3y+5z = 9,\ 7x+3y-2z = 8,\ 2x+3y+\lambda z = \mu$ may have
(a) Unique solution (b) Many solution (c) No solution.

20. Show that the equations, $-2x+y+z = a,\ x-2y+z = b,$
$x+y-2z = c$ is consistent if $a+b+c = 0$. Solve the system of equations when $(a, b, c) = (1, 1, -2)$.

ANSWERS

1. 3	2. 3	3. 2	4. 2	5. 3
6. 3	7. 3	8. 2	9. 3	10. 2

11. $x = 1,\ y = 3,\ z = 5$
12. $x = -1, y = 0, z = 2$
13. $x = 3,\ y = 2,\ z = 1$
14. $x = -10k/7,\ \ y = 8k/7,\ \ z = k$
15. $x = 1,\ y = 3k-2,\ z = k$
16. $x = k-2,\ \ y = 3-2k,\ \ z = k$
17. Inconsistent
18. Inconsistent
19. (a) Unique solution : $\lambda \neq 5$,
 (b) Infinite solution : $\lambda = 5,\ \ \mu = 9$
 (c) No solution : $\lambda = 5,\ \ \mu \neq 9$
20. $x = y = k-1,\ \ z = k$

Solution of a system of non homogeneous equations

7.7.1 Gauss elimination method

This method is illustrated by considering a system of three independent equations in three unknowns. The method is very much similar to the method we employed in solving a system of equations by testing its consistency.

Consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \qquad \ldots \text{(i)}$$

The system (i) is equivalent to the matrix equation

$$AX = B \qquad \ldots \text{(ii)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The method aims in reducing the coefficient matrix A to an upper triangular matrix.

We consider the augmented matrix $[A : B]$

We have $[A : B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$

Step - 1. We use the element $a_{11}\ (\neq 0)$ to make the elements a_{21} and a_{31} zero by elementary row transformations.

This transforms $[\,A : B\,]$ into the form

$$[\,A : B\,] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a_{22}' & a_{23}' & : & b_2' \\ 0 & a_{32}' & a_{33}' & : & b_3' \end{bmatrix} \qquad \ldots \text{(iii)}$$

Step - 2 We use the element $a_{22}'\ (\neq 0)$ to make the element a_{32}' zero by elementary row transformation.

This transforms $[\,A : B\,]$ as in (iii) into the form

$$[\,A : B\,] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a_{22}' & a_{23}' & : & b_2' \\ 0 & 0 & a_{33}'' & : & b_3'' \end{bmatrix} \qquad \ldots \text{(iv)}$$

(*Observe that the matrix A is reduced to the upper triangular form*)

From (iv), the given system of linear equations is equivalent to the system of equations

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{22}' x_2 + a_{23}' x_3 &= b_2' \\ a_{33}'' x_3 &= b_3'' \end{aligned}$$

We get x_3 from the last of the equation and by back substitution we get x_2 and x_1 The values of x_1, x_2, x_3 so obtained constitutes the exact solution of the given system of equations.

Note : *In case a_{11} or a_{22}' is zero we only need to re-arrange the equations.*

This method can be regarded as the modification of Gauss elimination method.

This method aims in reducing the coefficient matrix A to a diagonal matrix.

Step-1 This step is same as in Gauss elimination method.

Step-2 Referring to (iii), we use the element $a_{22}'\ (\neq 0)$ to make the elements a_{12} and a_{32}' zero by elementary row transformations.

With this transformation we obtain

$$[A:B] \sim \begin{bmatrix} a_{11} & 0 & a_{13}' & : & b_1' \\ 0 & a_{22}' & a_{23}' & : & b_2' \\ 0 & 0 & a_{33}'' & : & b_3'' \end{bmatrix} \qquad \ldots \text{(v)}$$

Step-3 Finally we use the element $a_{33}''\ (\neq 0)$ to make the elements a_{23}', a_{13}' zero by elementary row transformations.

With this transformation we obtain

$$[A:B] \sim \begin{bmatrix} a_{11} & 0 & 0 & : & b_1'' \\ 0 & a_{22}' & 0 & : & b_2'' \\ 0 & 0 & a_{33}'' & : & b_3'' \end{bmatrix} \qquad \ldots \text{(vi)}$$

(Observe that the matrix A is reduced to the diagonal form)

It can be easily seen from (vi) that

$$a_{11}\, x_1 = b_1'',\ a_{22}'\, x_2 = b_2'',\ a_{33}''\, x_3 = b_3''$$

Thus we get the required x_1, x_2, x_3 being the exact solution.

>> **(a)** *By Gauss elimination method :*

The augmented matrix of the system is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 1 & -2 & 3 & : & 8 \\ 2 & 1 & -1 & : & 3 \end{bmatrix}$$

$$R_2 \to -R_1 + R_2,\ R_3 \to -2R_1 + R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & -1 & -3 & : & -15 \end{bmatrix}$$

$$R_3 \to R_2 + (-3)\, R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 11 & : & 44 \end{bmatrix}$$

Hence we have, $x \quad +y \quad +z \quad = 9$

$$-3y \quad +2z \quad = -1$$

$$11z \quad = 44 \qquad \therefore \; z = 4$$

By back substitution, $-3y+8 = -1 \quad \therefore \; y = 3$; Also $x = 2$

Thus $x = 2, \; y = 3, \; z = 4$ is the required solution.

(b) *By Gauss Jordan method :*
The first step is same as in Gauss elimination method. So we have

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & -1 & -3 & : & -15 \end{bmatrix}$$

We use the leading non zero entry in second row (– 3) to make the element above (1) and below (– 1) zero.

$$R_1 \to R_2 + 3R_1, \quad R_3 \to R_2 + (-3)\,R_3$$

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 5 & : & 26 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 11 & : & 44 \end{bmatrix}$$

$R_3 \to 1/11 \cdot R_3$ (*This step is performed only for convenience to get 1 as the leading entry in the third row*).

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 5 & : & 26 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

[*We use the element 1 in the third row 1 to make the two elements above (2 and 5) zero.*]

$$R_1 \to -5R_3 + R_1, \quad R_2 \to -2R_3 + R_2$$

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 0 & : & 6 \\ 0 & -3 & 0 & : & -9 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

Hence we have $3x = 6, \; -3y = -9, \; z = 4$

Thus $\quad x = 2, \; y = 3, \; z = 4$ is the required solution.

34. *Solve by Gauss elimination method*

[illegible]

$$4x_1 + 11x_2 - x_3 = 33$$

$$8x_1 - 3x_2 + 2x_3 = 20$$

>> The augmented matrix of the system is

$$[A:B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

$$R_2 \to -2R_1 + R_2, \quad R_3 \to -4R_1 + R_3$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix}$$

$$1/9 \cdot R_2, \quad -1/7 \cdot R_3$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & -1 & : & 1 \\ 0 & 1 & 2 & : & 4 \end{bmatrix}$$

$$R_3 \to -R_2 + R_3$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & 3 & : & 3 \end{bmatrix}$$

Hence we have, $2x + y + 4z = 12$

$$y - z = 1$$

$$3z = 3 \quad \therefore \; z = 1$$

By back substitution, $y = 2$ and $x = 3$

Thus $\mathbf{x = 3,\ y = 2,\ z = 1}$ is the required solution.

35. *Solve by Gau* [illegible]

>> *It is convenient to perform row transformations if the leading entry is 1 We shall write the augmented matrix by interchanging the first equation with the fourth equation. Fourth equation has three of its coefficients 1.*

$$[A:B] = \left[\begin{array}{cccc:c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right]$$

$R_2 \to -R_1 + R_2,\ R_3 \to -R_1 + R_3,\ \ R_4 \to -5R_1 + R_4$

$$[A:B] \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 4 & 6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right]$$

$1/3 \cdot R_2$

$$[A:B] \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 4 & 6 \\ 0 & 2 & 0 & -1 & 6 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right]$$

$R_4 \to 2R_2 + R_4$

$$[A:B] \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 4 & 6 \\ 0 & 2 & 0 & -1 & 6 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right]$$

$R_4 \to 4R_3 + 5R_4$

$$[A:B] \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 4 & 6 \\ 0 & 2 & 0 & -1 & 6 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & -117 & 234 \end{array}\right]$$

Hence we have, $x_1 + x_2 + x_3 + 4x_4 = -6$

$$2x_2 + 0x_3 - x_4 = 6$$

$$5x_3 - 3x_4 = 1$$

$$-117x_4 = 234 \quad \therefore\ x_4 = -2$$

By back substitution we get $x_3 = -1,\ x_2 = 2,\ x_1 = 1$

Thus $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2$ is the required solution.

36. *Apply Gauss - Jordan method to solve the system of equations*

$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$
$$x + y + z = 9$$

>> As it is convenient to have the leading coefficient as 1 we shall interchange the first and third equations. The augmented matrix will be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$$R_2 \to -2R_1 + R_2, \quad R_3 \to -2R_1 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$$R_1 \to R_2 + R_1, \quad R_3 \to 3R_2 + R_3$$

$$[A \cdot B] \sim \begin{bmatrix} 1 & 0 & -2 & & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & & -20 \end{bmatrix}$$

$$-1/4 \cdot R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & . & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_1 \to 2R_3 + R_1, \quad R_2 \to 3R_3 + R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Hence we have $x = 1, \; -y = -3, \; z = 5$

Thus $x = 1, \; y = 3, \; z = 5$ is the required solution

37.

>> The augmented matrix associated with the given system is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 2 & -1 & 2 & -1 & : & -5 \\ 3 & 2 & 3 & 4 & : & 7 \\ 1 & -2 & -3 & 2 & : & 5 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ R_3 \to -3R_1 + R_3,\ R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & -3 & 0 & -3 & : & -9 \\ 0 & -1 & 0 & 1 & : & 1 \\ 0 & -3 & -4 & 1 & : & 3 \end{bmatrix}$$

$-1/3 \cdot R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & 0 & 1 & : & 3 \\ 0 & -1 & 0 & 1 & : & 1 \\ 0 & -3 & -4 & 1 & : & 3 \end{bmatrix}$$

$R_1 \to -R_2 + R_1,\ R_3 \to R_2 + R_3,\ R_4 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 1 & 0 & : & -1 \\ 0 & 1 & 0 & 1 & : & 3 \\ 0 & 0 & 0 & 2 & : & 4 \\ 0 & 0 & -4 & 4 & : & 12 \end{bmatrix}$$

We shall perform $1/2 \cdot R_3$, $1/4 \cdot R_4$ and then interchange the rows R_3 and R_4

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 1 & 0 & : & -1 \\ 0 & 1 & 0 & 1 & : & 3 \\ 0 & 0 & -1 & 1 & : & 3 \\ 0 & 0 & 0 & 1 & : & 2 \end{bmatrix}$$

$R_2 \to -R_4 + R_2,\ R_3 \to -R_4 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 1 & 0 & : & -1 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & -1 & 0 & : & 1 \\ 0 & 0 & 0 & 1 & : & 2 \end{bmatrix}$$

$R_1 \to R_3 + R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & -1 & 0 & : & 1 \\ 0 & 0 & 0 & 1 & : & 2 \end{bmatrix}$$

Hence we have $x_1 = 0,\ x_2 = 1,\ -x_3 = 1,\ x_4 = 2$

Thus $\quad x_1 = 0,\ x_2 = 1,\ x_3 = -1,\ x_4 = 2$ is the required solution.

38. *Solve by Gauss elimination method*

[illegible]

>> The augmented matrix associated with the given system of equations is

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 3 & -1 & 4 & : & 4 \\ 2 & 1 & -2 & : & 5 \end{bmatrix}$$

$R_2 \to -3R_1 + R_2,\ R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 0 & 5 & -5 & : & -2 \\ 0 & 5 & -8 & : & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 0 & 5 & -5 & : & -2 \\ 0 & 0 & -3 & : & 3 \end{bmatrix}$$

Hence we have, $\quad x_1 - 2x_2 + 3x_3 = 2$

$$5x_2 - 5x_3 = -2$$

$$-3x_3 = 3 \quad \therefore \quad x_3 = -1$$

By back substitution $5x_2 + 5 = -2 \ \therefore \ x_2 = -7/5 = -1.4$

Also, $x_1 - 2(-1.4) - 3 = 2 \ \therefore \ x_1 = 2.2$

Thus $\quad x_1 = 2.2,\ x_2 = -1.4,\ x_3 = -1$ is the required solution.

39. *Apply Gauss elimination method to solve the system of equations*

$$2x + 5y + 7 \text{[illegible]}$$
$$2x + y - z = 0$$
$$x + y + z = 9$$

>> As it is convenient to have the leading coefficient as 1 we shall interchange the first and third equations. The augmented matrix will be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ \ R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_3 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

Hence we have,
$$x + y + z = 9$$
$$-y - 3z = -18$$
$$-4z = -20 \quad \therefore\ z = 5$$

By back substitution $y = 3$ and $x = 1$

Thus $x = 1,\ y = 3,\ z = 5$ is the required solution.

40. *Solve the following system of equations by Gauss elimination method*

$$x_1 + x_2 + x_3 + x_4 = 2$$

[illegible]

>> The augmented matrix associated with the given system is

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 2 & -1 & 2 & -1 & : & -5 \\ 3 & 2 & 3 & 4 & : & 7 \\ 1 & 2 & -3 & 2 & & 5 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ \ R_3 \to -3R_1 + R_3,\ \ R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & -3 & 0 & -3 & : & -9 \\ 0 & -1 & 0 & 1 & : & 1 \\ 0 & -3 & -4 & 1 & : & 3 \end{bmatrix}$$

$R_3 \to 1/3 \cdot R_2 + R_3\,,\ R_4 \to -R_2 + R_4$

$$[A:B] \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$[A:B] \sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & 0 & -4 & 4 & 12 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

Hence we have,
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ -3x_2 + 0x_3 - 3x_4 &= -9 \\ -4x_3 + 4x_4 &= 12 \\ 2x_4 &= 4 \qquad \therefore\ x_4 = 2 \end{aligned}$$

Further by back substitution we get $x_3 = -1\,,\ x_2 = 1$ and $x_1 = 0$

Thus $x_1 = 0\,,\ x_2 = 1\,,\ x_3 = -1\,,\ x_4 = 2$ is the required solution.

Note : *In many practical situations we come across with more number of equations involving decimal quantities also. In such cases we adopt sequential steps which can be programmed, as illustrated in the following problem. We donot express the equations in the matrix form also.*

An Illustrative Example on Gauss elimination method.

$$\begin{aligned} 0.2\,x + 0.3\,y - 0.1\,z &= 0.5 \\ 0.4\,x + 0.4\,y - 0.3\,z &= 0.3 \\ 0.2\,x - 0.3\,y + 0.2\,z &= 0.2 \end{aligned}$$

>> **Step-1** The coefficient of x in all the three equations are made 1 by dividing the equations respectively by 0.2, 0.4 and 0.2. We get,

$$\begin{aligned} x + 1.5\,y - 0.5\,z &= 2.5 \\ x + y - 0.75\,z &= 0.75 \\ x - 1.5\,y + z &= 1 \end{aligned}$$

Step-2 We subtract the second and third equations from the first equation. We get,

$$\begin{aligned} x + 1.5\,y - 0.5\,z &= 2.5 \\ 0.5\,y + 0.25\,z &= 1.75 \\ 3\,y - 1.5\,z &= 1.5 \end{aligned}$$

Step-3 The coefficient of y in the second and third equations are made 1 by dividing the equations respectively by 0.5 and 3. We get,

$$x + 1.5\,y - 0.5\,z = 2.5$$

$$y + 0.5\,z = 3.5$$

$$y - 0.5\,z = 0.5$$

Step-4 We subtract the third equation from the second equation. We get

$$\begin{aligned} x + 1.5\,y - 0.5\,z &= 2.5 \\ y + 0.5\,z &= 3.5 \\ z &= 3 \end{aligned}$$

By back substitution we get $y = 2$ and $x = 1$

Thus $\boldsymbol{x = 1,\ y = 2,\ z = 3}$ is the required solution.

Remark : *In the event of getting more decimals in the process of division, we can fix the constants to suitable/desired number of decimal places.*

EXERCISES

Solve the following system of equations by Gauss elimination method and also by Gauss - Jordan method.

1. $x + 2y + z = 3\,,\ 2x + 3y + 3z = 10\,,\ 3x - y + 2z = 13$
2. $2x + 3y - z = 5\,,\ 4x + 4y - 3z = 3\,,\ 2x - 3y + 2z = 2$
3. $3x + 4y + 5z = 18\,,\ 2x - y + 8z = 13\,,\ 5x - 2y + 7z = 20$
4. $2x_1 - x_2 + x_3 = -1\,,\ 2x_2 - x_3 + x_4 = 1$

   $x_1 + 2x_3 - x_4 = -1\,,\ x_1 + x_2 + 2x_4 = 3$
5. $5x_1 + x_2 + x_3 + x_4 = 4\,,\ x_1 + 7x_2 + x_3 + x_4 = 12$

   $x_1 + x_2 + 6x_3 + x_4 = -5\,,\ x_1 + x_2 + x_3 + 4x_4 = -6$

ANSWERS

Note : *Answers are given in the respective order of the variables as in the given equations*

1. 2, −1, 3 **2.** 1, 2, 3 **3.** 3, 1, 1

4. −1, 0, 1, 2 **5.** 1, 2, −1, −2

Unit - VIII

LINEAR ALGEBRA - 2

8.1 Introduction

In this unit, we continue to discuss a few more matrix oriented concepts such as eigen values and eigen vectors of a square matrix, quadratic form expressible in the matrix form where we see the association with the eigen values and eigen vectors.

8.2 Linear Transformations

Transformation means change.

For example, the reader is familiar with the transformation from cartesian system to polar system The associated transformation is $x = r\cos\theta$ and $y = r\sin\theta$. Here (x, y) are cartesian coordinates and (r, θ) is expressible in terms of (x, y) as we have, $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$ which being the inverse transformation

A Linear Transformation in two dimensions is represented by

$$\left.\begin{aligned} y_1 &= a_1\, x_1 + a_2\, x_2 \\ y_2 &= b_1\, x_1 + b_2\, x_2 \end{aligned}\right\} \qquad \ldots (1)$$

This can be represented in the matrix form.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \ldots (2)$$

Similarly a linear transformation in three dimensions along with its matrix form is as follows.

$$\left.\begin{aligned} y_1 &= a_1\, x_1 + a_2\, x_2 + a_3\, x_3 \\ y_2 &= b_1\, x_1 + b_2\, x_2 + b_3\, x_3 \\ y_3 &= c_1\, x_1 + c_2\, x_2 + c_3\, x_3 \end{aligned}\right\} \qquad \ldots (3)$$

or $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \ldots (4)$$

We can as well write (2) and (4) in the form

$$Y = AX \qquad \ldots (5)$$

where Y , A , X are the associated matrices A is called the ***Transformation Matrix.***

Further if the matrix A is non singular ($|A| \neq 0$) then $Y = AX$ is called a *non singular transformation or regular transformation.* In this case

$$X = A^{-1} Y \qquad \ldots (6)$$

is called the *inverse transformation.*

Also if $|A| = 0$, the transformation $Y = AX$ is called a *singular transformation*

Next, let $Z = BY$ also be a linear transformation. Then we have,

$$Z = (BY) = B(AX) = (BA)X = CX \text{ (say) } \quad \text{where } C = BA$$

Here $Z = CX$ is called a *composite linear transformation.*

WORKED PROBLEMS

1. *If $\alpha = x\cos\theta - y\sin\theta$ and $\beta = x\sin\theta + y\cos\theta$ write the* [illegible] *transformation and prove that $A^{-1} = A'$ Also write the inverse transformation*

>> We have, $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $Y = AX$ is the associated matrix representation of the given linear transformation.

$$|A| = \cos^2\theta + \sin^2\theta = 1$$

$$Adj A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \ldots (1)$$

Also $\quad A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \ldots (2)$

From (1) and (2), $\mathbf{A^{-1} = A'}$

(**Remark** : *This implies that A is an orthogonal matrix*)

The given transformation is $Y = AX$ Hence the associated inverse transformation is,

$$X = A^{-1} Y$$

ie., $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Thus the inverse transformation is, $x = \alpha\cos\theta + \beta\sin\theta$ **and** $y = -\alpha\sin\theta + \beta\cos\theta$

2. *Find the inverse transformation of the following linear transformation*

$$y_1 = x_1 + 2x_2 + 5x_3$$
$$y_2 = 2x_1 + 4x_2 + 11x_3$$
$$y_3 = -x_2 + 2x_3$$

The given linear transformation in the matrix form is,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

ie., $Y = AX$ where we have,

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{bmatrix}$$

We compute, $A^{-1} = \dfrac{1}{|A|}\,(Adj\,A)$

$$|A| = 1(8+11) - 2(4-0) + 5(-2-0) = 1$$

$$Adj\,A = \begin{bmatrix} +(8+11), & -(4+5), & +(22-20) \\ -(4-0), & +(2-0), & -(11-10) \\ +(-2-0), & -(-1-0), & +(4-4) \end{bmatrix}' = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

Since $|A| = 1$, $A^{-1} = Adj\,A$ itself.

Inverse transformation is given by $X = A^{-1}Y$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Thus, $x_1 = 19y_1 - 9y_2 + 2y_3$, $x_2 = -4y_1 + 2y_2 - y_3$, $x_3 = -2y_1 + y_2$ **is the required inverse transformation.**

3. *Show that the transformation* $y_1 = 2x_1 - 2x_2 - x_3$, $y_2 = -4x_1 + 5x_2 + 3x_3$, $y_3 = x_1 - x_2 - x_3$ *is regular and find the inverse transformation.*

>> The given transformation in the matrix form is,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

ie., $Y = AX$, where

$$A = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Now, $|A| = 2(-5+3)+2(4-3)-1(4-5) = -1$

$|A| = -1 \neq 0 \Rightarrow$ **the transformation is regular.**

We compute $A^{-1} = \dfrac{1}{|A|}(Adj\,A)$

$$Adj\,A = \begin{bmatrix} +(-5+3), & -(2-1), & +(-6+5) \\ -(4-3), & +(-2+1), & -(6-4) \\ +(4-5), & -(-2+2), & +(10-8) \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$$

Hence $A^{-1} = -\begin{bmatrix} -2 & -1 & -1 \\ -1 & -1 & -2 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

Inverse transformation is given by $X = A^{-1}Y$

ie.,
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Thus, $x_1 = 2y_1 + y_2 + y_3$, $x_2 = y_1 + y_2 + 2y_3$, $x_3 = y_1 - 2y_3$ **is the required inverse transformation.**

4. *Represent each of the transformation* [illegible] $x_2 = -y_1 + 4y_2$, $y_2 = 3z_1$ [illegible] *expresses* x_1, x_2 *in term* [illegible]

>> We have by data,

$$x_1 = 3y_1 + 2y_2, \quad x_2 = -y_1 + 4y_2 \qquad \ldots (1)$$

$$y_1 = z_1 + 2z_2, \quad y_2 = 3z_1 \qquad \ldots (2)$$

The matrix representation of (1) and (2) are as follows.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

ie., $\quad X = AY$ and $Y = BZ$

where $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Now we have, $X = AY = A(BZ) = (AB)Z$

ie., $\quad X = (AB)Z$ is the composite transformation.

Further, $AB = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3+6, & 6+0 \\ -1+12, & -2+0 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & 2 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Thus $\mathbf{x_1 = 9z_1 + 6z_2}$ **&** $\mathbf{x_2 = 11z_1 - 2z_2}$ *is the required composite transformation*

5. *Given the linear transformations* $y_1 = 5x_1 + 3x_2 + 3x_3$, $y_2 = 3x_1 + 2x_2 - 2x_3$, $y_3 = 2x_1 - x_2 + 2x_3$ *and* $z_1 = 4x_1 + 2x_3$, $z_2 = x_2 + 4x_3$, $z_3 = 5x_3$ *establish the linear transformation from* z_1, z_2, z_3 *to* y_1, y_2, y_3 *by matrix approach*

>> We have by data,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \ldots (1)$$

and
$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \ldots (2)$$

The equivalent form of (1) and (2) are, $Y = AX$ and $Z = BX$

where $A = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

$$Z = BX \Rightarrow X = B^{-1} Z \text{ and hence,}$$

$$Y = AX = A(B^{-1}Z) = (AB^{-1})Z$$

We need to compute B^{-1} and the matrix prouct AB^{-1}

$$B^{-1} = \frac{1}{|B|} \, AdjB$$

We have $|B| = 20$

$$Adj\ B = \begin{bmatrix} +(5-0), & -(0-0), & +(0-2) \\ -(0-0), & +(20-0), & -(16-0) \\ +(0-0), & -(0-0), & +(4-0) \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{1}{20} \begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$$

Next, $AB^{-1} = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \cdot \frac{1}{20} \begin{bmatrix} 5 & 0 & -2 \\ 0 & 20 & -16 \\ 0 & 0 & 4 \end{bmatrix}$

ie., $AB^{-1} = \frac{1}{20} \begin{bmatrix} 25+0+0, & 0+60+0, & -10-48+12 \\ 15+0+0, & 0+40+0, & -6-32-8 \\ 10+0+0, & 0-20+0, & -4+16+8 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 25 & 60 & -46 \\ 15 & 40 & -46 \\ 10 & -20 & 20 \end{bmatrix}$

We have $Y = (AB^{-1})Z$

ie., $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 25 & 60 & -46 \\ 15 & 40 & -46 \\ 10 & -20 & 20 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Thus, $y_1 = (5/4) z_1 + 3 z_2 - (23/10) z_3$

$y_2 = (3/4) z_1 + 2 z_2 - (23/10) z_3$

$y_3 = (1/2) z_1 - z_2 + z_3$ is the required linear transformation

8.3 Eigen values and Eigen vectors of a square matrix

Definition Given a square matrix A, if there exists a scalar λ (*real or complex*) and a non zero column matrix X such that $AX = \lambda X$, then λ is called an ***eigen value*** of A and X is called an ***eigen vector*** of A corresponding to an eigen value λ.

If I is the unit matrix of the same order as that of A, we have $X = IX$ and hence $AX = \lambda X$ can be written as

$$AX = \lambda(IX) = (\lambda I)X$$

i.e., $[A - \lambda I][X] = [0]$, $[0]$ is the null matrix.

Let us consider a square matrix of order 3 represented by

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{Also} \quad \lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{bmatrix} \quad \text{Also let } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It can be easily seen that $[A - \lambda I][X] = [0]$ represents a set of homogeneous equations in 3 unknowns.

i.e.,

$$\begin{aligned} (a_1 - \lambda)x + a_2 y + a_3 z &= 0 \\ b_1 x + (b_2 - \lambda)y + b_3 z &= 0 \\ c_1 x + c_2 y + (c_3 - \lambda)z &= 0 \end{aligned}$$

A nontrivial solution for this system exists if the determinant of the coefficient matrix is zero

i.e.,

$$\begin{vmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{vmatrix} = 0$$

On expanding we get a cubic equation in λ which is called the ***characteristic equation*** of A. The roots of this equation are the *eigen values* which are also called *eigen roots* or *characteristic roots* or *latent roots*. For each value of λ there will be an eigen vector $X \neq 0$ which is also called a *characteristic vector*.

[8.3] Properties of eigen values and eigen vectors

1. Sum of the eigen values of a square matrix is equal to the 'trace' (sum of th principal diagonal elements) of the matrix.
2. Product of the eigen values of a square matrix is equal to the determinant of th matrix.
3. If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigen values of an n^{th} order square matrix A, the $\lambda_1^k, \lambda_2^k, \lambda_3^k, \cdots \lambda_n^k$ are the eigen values of the matrix A^k.
4. The eigen vector X of a matrix is not unique.
5. If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the distinct eigen values of an n^{th} order square matrix A, the the corresponding eigen vectors $X_1, X_2, \cdots, X_n$ form a linearly independe set.
6. If two or more eigen values are equal it may or may not be possible to get linearl independent eigen vectors corresponding to the coincident eigen values
7. If $X_1 = (x_1, y_1, z_1)$, $X_2 = (x_2, y_2, z_2)$ then X_1, X_2 are called orthogon vectors if,

   $$X_1 \cdot X_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = 0$$

8. Eigen vectors of a symmetric matrix corresponding to different eigen values a orthogonal.
9. 'Norm' of a vector $X = (x, y, z)$ denoted by $\| x \|$ is equal t $\sqrt{x^2+y^2+z^2} = k$ (say). Then $\left(\frac{x}{k}, \frac{y}{k}, \frac{z}{k}\right)$ is called the normalized vector
10. The matrix $P = \left[\frac{X_1'}{\| X_1 \|}, \frac{X_2'}{\| X_2 \|}, \frac{X_3'}{\| X_3 \|} \right]$ will be an orthogon matrix.

Working procedure for problems

- Given a square matrix A (*usually of order* 3) we form $| A - \lambda I | = 0$. On expanding we get the characteristic equation of A. By solving it we get all th eigen values.
- We then form the system of homogeneous equations from the matrix equation $[A - \lambda I][X] = [0]$ and solve for (x, y, z) corresponding to every value of λ
- Simple techniques of solving or the rule of cross multiplication (*for any pair equations*) can be employed The values x, y, z obtained by the rule of cro multiplication satisfy simultaneously all the three equations.

WORKED PROBLEMS

6. Find all the eigen values and the corresponding eigen vectors of the matrix

>> The characteristic equation of A is $|A-\lambda I| = 0$

i.e., $$\begin{vmatrix} (8-\lambda) & -6 & 2 \\ -6 & (7-\lambda) & -4 \\ 2 & -4 & (3-\lambda) \end{vmatrix} = 0$$

On expanding we have,

$$(8-\lambda)\,[(7-\lambda)(3-\lambda)-16]+6\,[-6(3-\lambda)+8]+2\,[24-2(7-\lambda)] = 0$$

i.e., $$(8-\lambda)\,[5-10\lambda+\lambda^2]+6\,[6\lambda-10]+2\,[10+2\lambda] = 0$$

i.e., $-\lambda^3+18\lambda^2-45\lambda = 0$, on simplification.

or $$\lambda^3-18\lambda^2+45\lambda = 0$$

i.e., $$\lambda\,(\lambda^2-18\lambda+45) = 0 \quad\text{or}\quad \lambda(\lambda-3)(\lambda-15) = 0$$

$\therefore$ $\lambda = 0,\ 3,\ 15$ are the eigen values of A.

We now form the system of equations

$$\begin{aligned} (8-\lambda)x \quad -6y \quad\quad +2z \quad &= 0 \\ -6x \quad +(7-\lambda)y-4z \quad &= 0 \\ 2x \quad -4y \quad +(3-\lambda)z &= 0 \end{aligned} \qquad \ldots(1)$$

Case i Let $\lambda = 0$. The system of equations become

$$8x-6y+2z = 0 \qquad \ldots(\text{i})$$

$$-6x+7y-4z = 0 \qquad \ldots(\text{ii})$$

$$2x-4y+3z = 0 \qquad \ldots(\text{iii})$$

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

i.e., $$\frac{x}{10} = \frac{y}{20} = \frac{z}{20} \quad\text{or}\quad \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$\therefore$ (x, y, z) are proportional to $(1, 2, 2)$ and we can write $x = k, y = 2k, z = 2k$ where k is arbitrary. However it is enough to keep the values of (x, y, z) in the simplest form $x = 1, y = 2, z = 2$. These values satisfy all the equations simultaneously.

Thus the eigen vector X_1 corresponding to the eigen value $\lambda = 0$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

[*The same will be written as a row vector in the form* $X_1 = (1, 2, 2)$ *in future*]

Thus $X_1 = (1, 2, 2)$ is the eigen vector corresponding to $\lambda = 0$

Case - ii : Let $\lambda = 3$ and the corresponding equations from (1) are

$$5x - 6y + 2z = 0 \qquad \ldots \text{(iv)}$$

$$-6x + 4y - 4z = 0 \qquad \ldots \text{(v)}$$

$$2x - 4y + 0z = 0 \qquad \ldots \text{(vi)}$$

From (iv) and (v) we have as before,

$$\frac{x}{24 - 8} = \frac{-y}{-20 + 12} = \frac{z}{20 - 36} \quad \text{or} \quad \frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16}$$

i.e., $\quad \dfrac{x}{2} = \dfrac{y}{1} = \dfrac{z}{-2} \quad \therefore \quad (x, y, z) = (2, 1, -2)$

Thus $\quad X_2 = (2, 1, -2)$ is the eigen vector corresponding to $\lambda = 3$.

Case - iii : Let $\lambda = 15$ and the associated equations from (1) are

$$-7x - 6y + 2z = 0 \qquad \ldots \text{(vii)}$$

$$-6x - 8y - 4z = 0 \qquad \ldots \text{(viii)}$$

$$2x - 4y - 12z = 0 \qquad \ldots \text{(ix)}$$

From (vii) and (viii) we have,

$$\frac{x}{24 + 16} = \frac{-y}{28 + 12} = \frac{z}{56 - 36} \quad \text{or} \quad \frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$$

i.e., $\quad \dfrac{x}{2} = \dfrac{y}{-2} = \dfrac{z}{1} \quad \therefore \quad (x, y, z) = (2, -2, 1)$

Thus $\quad X_3 = (2, -2, 1)$ is the eigen vector corresponding to $\lambda = 15$

Note . *The characteristic equation of a third order square matrix A can be obtained without expanding* $| A - \lambda I | = 0$ *by the following rule :*

$$\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma m_d)\lambda - | A | = 0, \text{ where}$$

Σd = Sum of the diagonal elements of A

Σm_d = Sum of the minors of the diagonal elements of A

$| A |$ = Determinant of A

In the Example - 6

$$\Sigma d = 8 + 7 + 3 = 18$$

$$\Sigma m_d = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$$

$$| A | = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 40 - 60 + 20 = 0$$

Substituting in the rule we get the characteristic equation,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

Remark . *Observe the verification of some of the properties connected with eigen values and eigen vectors stated earlier.*

(i) Sum of all the eigen values $= 0 + 3 + 15 = 18$ is equal to the trace of A which being $8 + 7 + 3 = 18$

(ii) Product of all the eigen values $= 0$. $| A | = 0$

(iii) $X_1 \cdot X_2 = 2 + 2 - 4 = 0$, $X_2 \cdot X_3 = 4 - 2 - 2 = 0$, $X_3 \cdot X_1 = 2 - 4 + 2 = 0$
$\Rightarrow$ Eigen vectors X_1, X_2, X_3 are orthogonal.

(iv) $\| X_1 \| = \sqrt{1 + 4 + 4} = 3$, $\| X_2 \| = \sqrt{4 + 1 + 4} = 3$, $\| X_3 \| = \sqrt{4 + 4 + 1} =$
Normalized eigen vectors of X_1, X_2, X_3 are respectively $(1/3, 2/3, 2/3)$, $(2/3, 1/3, -2/3)$, $(2/3, -2/3, 1/3)$

The matrix $P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is an orthogonal matrix.

(*PP' = I can be easily verified*).

>> $| A - \lambda I | = 0$ is the characteristic equation of A .

i.e., $\begin{vmatrix} (7-\lambda) & -2 & 0 \\ -2 & (6-\lambda) & -2 \\ 0 & -2 & (5-\lambda) \end{vmatrix} = 0$

or $(7-\lambda)\,[(6-\lambda)(5-\lambda)-4]+2\,[-2(5-\lambda)] = 0$

i.e., $-\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0$, on simplification.

or $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

To solve this cubic equation we shall first find a root by inspection by simply trying values for $\lambda = 1, 2, 3, \ldots$ *(If λ is negative all the terms of the equation will be negative and hence cannot become zero)*

Putting $\lambda = 3$ we have $27 - 162 + 297 - 162 = 324 - 324 = 0$

Thus $\lambda = 3$ is a root by inspection. The other two roots can be found by synthetic division as follows.

$$\begin{array}{c|cccc} 3 & 1 & -18 & 99 & -162 \\ & 0 & +3 & -45 & +162 \\ \hline & 1 & -15 & +54 & 0 \end{array}$$

$\therefore$ the quadratic is $\lambda^2 - 15\lambda + 54 = 0$

i.e., $(\lambda-6)(\lambda-9) = 0$ or $\lambda = 6, 9$

Thus **$\lambda = 3, 6, 9$ are the eigen values.**

We now form the system of equations

$$\begin{aligned} (7-\lambda)x \quad -2y \quad &+0z = 0 \\ -2x \quad +(6-\lambda)y &-2z = 0 \\ 0x \quad -2y \quad &+(5-\lambda)z = 0 \end{aligned} \qquad \ldots (1)$$

Case 1. Let $\lambda = 3$ and the corresponding equations are

$4x - 2y + 0z = 0$ ... (i)

$-2x + 3y - 2z = 0$ ... (ii)

$0x - 2y + 2z = 0$ ... (iii)

From (i) and (ii) we have by applying the rule of cross multiplication,

$$\frac{x}{4-0} = \frac{-y}{-8-0} = \frac{z}{12-4} \text{ or } \frac{x}{4} = \frac{y}{8} = \frac{z}{8} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$\therefore$ $X_1 = (1, 2, 2)$ is the eigen vector corresponding to $\lambda = 3$.

Case - ii : Let $\lambda = 6$ and the corresponding equations from (1) are

$$1x - 2y + 0z = 0 \quad \ldots \text{(iv)}$$
$$-2x - 0y - 2z = 0 \quad \ldots \text{(v)}$$
$$0x - 2y - 1z = 0 \quad \ldots \text{(vi)}$$

From (iv) and (v), $\frac{x}{4} = \frac{-y}{-2} = \frac{z}{-4}$ or $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$

$\therefore$ $X_2 = (2, 1, -2)$ is the eigen vector corresponding to $\lambda = 6$.

Case - iii : Let $\lambda = 9$ and the corresponding equations from (1) are

$$-2x - 2y + 0z = 0 \quad \ldots \text{(vii)}$$
$$-2x - 3y - 2z = 0 \quad \ldots \text{(viii)}$$
$$0x - 2y - 4z = 0 \quad \ldots \text{(ix)}$$

From (vii) and (viii), $\frac{x}{4} = \frac{-y}{4} = \frac{z}{2}$ or $\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$

$\therefore$ $X_3 = (2, -2, 1)$ is the eigen vector corresponding to $\lambda = 9$.

>> The characteristic equation of the given matrix is

$$\begin{vmatrix} (2-\lambda) & 0 & 1 \\ 0 & (2-\lambda) & 0 \\ 1 & 0 & (2-\lambda) \end{vmatrix} = 0$$

i.e., $(2-\lambda)(2-\lambda)^2 - (2-\lambda) = 0$

i.e., $(2-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$

or $(2-\lambda)(2-\lambda+1)(2-\lambda-1) = 0$

i.e., $(2-\lambda)(3-\lambda)(1-\lambda) = 0$ or $\lambda = 2, \lambda = 3, \lambda = 1$

Thus $\lambda = 1, 2, 3$ are the characteristic roots.

Let us now form the system of equations

$$\begin{aligned}(2-\lambda)x \quad &+0y &+1z &= 0\\ 0x \quad &+(2-\lambda)y &+0z &= 0\\ x \quad &+0y &+(2-\lambda)z &= 0\end{aligned} \qquad \ldots(1)$$

Case - i : Let $\lambda = 1$ and the corresponding equations are

$$x+z=0, \quad y=0, \quad x+z=0$$

i.e., $x = -z$ and if $z = 1$ is arbitrarily chosen for convenience then $x = -1$

$\therefore \ (x, y, z) = (-1, 0, 1)$ [*The rule of cross multiplication is not used as the equations are highly simple*]

$\therefore \quad X_1 = (-1, 0, 1)$ is the eigen vector corresponding to $\lambda = 1$.

Case - ii Let $\lambda = 2$ and the corresponding equations from (1) are $z = 0$, $0 = 0$, $x = 0$ since $x = 0$ and $z = 0$, y can be chosen arbitrarily, say $y = 1$.

$\therefore \quad X_2 = (0, 1, 0)$ is the eigen vector corresponding to $\lambda = 2$.

Case - iii : Let $\lambda = 3$ and the corresponding equations from (1) are $-x+z=0$, $-y=0$, $x-z=0 \ \therefore \ x = z$ and $y = 0$

Let us choose $x = z = 1$ (*arbitrary*)

$\therefore \quad X_3 = (1, 0, 1)$ is the eigen vector corresponding to $\lambda = 3$.

>> $\quad |A - \lambda I| = 0$ is the characteristic equation of A

i.e., $\begin{vmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

i.e., $(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1-\lambda]=0$

i.e., $(-2-\lambda)(-\lambda+\lambda^2-12)+(4\lambda+12)+(9+3\lambda)=0$

i.e., $(-2-\lambda)(\lambda+3)(\lambda-4)+4(\lambda+3)+3(\lambda+3)=0$

i.e., $(\lambda+3)[(-2-\lambda)(\lambda-4)+4+3]=0$

i.e., $(\lambda+3)(-\lambda^2+2\lambda+15)=0$ or $(\lambda+3)(\lambda^2-2\lambda-15)=0$

i.e., $(\lambda+3)(\lambda+3)(\lambda-5)=0$ or $\lambda = -3, -3, 5$

$\therefore \quad \lambda_1 = -3, \ \lambda_2 = -3, \ \lambda_3 = 5$ are the eigen values.

We now form the system of equations.

$$\begin{aligned} (-2-\lambda)x + 2y \quad & -3z = 0 \\ 2x \quad + (1-\lambda)y & - 6z = 0 \\ -1x \quad - 2y \quad & -\lambda z = 0 \end{aligned} \qquad \ldots (1)$$

Case - i : Let $\lambda = -3$ and the corresponding equations are

$$x + 2y - 3z = 0 \qquad \ldots \text{(i)}$$

$$2x + 4y - 6z = 0 \qquad \ldots \text{(ii)}$$

$$-x + 2y + 3z = 0 \qquad \ldots \text{(iii)}$$

It should be observed that the equations (i), (ii), (iii) are all same and we have only one independent equation $x + 2y - 3z = 0$ (*In case the rule of cross multiplication is applied, we get $x = 0 = y = z$ which is a trivial solution*)

Two variables can be chosen arbitrarily.

Let $\quad z = k_1, \ y = k_2 \ \therefore \ x = 3k_1 - 2k_2$

Thus $\quad X_1 = (3k_1 - 2k_2, \ k_2, \ k_1)$ is the eigen vector corresponding to $\lambda = -3$ where $k_1, \ k_2$ are not simultaneously zero.

Case - ii : Let $\lambda = 5$ and the corresponding equations from (1) are

$$-7x + 2y - 3z = 0 \qquad \ldots \text{(iv)}$$

$$2x - 4y - 6z = 0 \qquad \ldots \text{(v)}$$

$$-1x - 2y - 5z = 0 \qquad \ldots \text{(vi)}$$

From (iv) and (v), $\dfrac{x}{-12-12} = \dfrac{-y}{42+6} = \dfrac{z}{28-4}$

i.e., $\quad \dfrac{x}{24} = \dfrac{-y}{48} = \dfrac{z}{24} \quad$ or $\quad \dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{-1}$

$\therefore \quad X_2 = (1, 2, -1)$ is the eigen vector corresponding to $\lambda = 5$.

>> The eigen values are obtained from the characteristic equation $|A - \lambda I| = 0$.

ie., $\begin{vmatrix} (1-\lambda) & 1 & 3 \\ 1 & (5-\lambda) & 1 \\ 3 & 1 & (1-\lambda) \end{vmatrix} = 0$

On expanding we get $\lambda^3 - 7\lambda^2 + 36 = 0$.

$\lambda = -2$ is a root by inspection. Now by synthetic division,

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$\Rightarrow \quad \lambda^2 - 9\lambda + 18 = 0$ or $(\lambda - 3)(\lambda - 6) = 0$ or $\lambda = 3, \lambda = 6$

$\therefore \quad \lambda = -2, 3, 6$ are the eigen values.

We now form the system of equations,

$$\left.\begin{array}{llll} (1-\lambda)x & +1y & +3z & = 0 \\ 1x & +(5-\lambda)y & +1z & = 0 \\ 3x & +1y & +(1-\lambda)z & = 0 \end{array}\right\} \quad \ldots (1)$$

Case (i) : Let $\lambda = -2$ and the corresponding equations are

$$\left.\begin{array}{l} 3x + 1y + 3z = 0 \\ 1x + 7y + z = 0 \\ 3x + 1y + 3z = 0 \end{array}\right\} \Rightarrow \frac{x}{-20} = \frac{-y}{0} = \frac{z}{20}$$

$\therefore \quad (x, y, z) = (1, 0, -1)$ is the eigen vector corresponding to $\lambda = -2$.

Case (ii) : Let $\lambda = 3$ and we have from (1),

$$\left.\begin{array}{l} -2x + 1y + 3z = 0 \\ 1x + 2y + 1z = 0 \\ 3x + 1y - 2z = 0 \end{array}\right\} \Rightarrow \frac{x}{-5} = \frac{y}{5} = \frac{z}{5}$$

$\therefore \quad (x, y, z) = (1, -1, 1)$ is the eigen vector corresponding to $\lambda = 3$

Case (iii) : Let $\lambda = 6$ and we have from (1),

$$\left.\begin{array}{l} -5x + 1y + 3z = 0 \\ 1x - 1y + 1z = 0 \\ 3x + 1y - 5z = 0 \end{array}\right\} \Rightarrow \frac{x}{4} = \frac{-y}{8} = \frac{z}{4} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$\therefore \quad (x, y, z) = (1, 2, 1)$ is the eigen vector corresponding to $\lambda = 6$.

Thus $-2, 3, 6$ are the eigen values and the corresponding eigen vectors are $(1, 0, -1)$; $(1, -1, 1)$; $(1, 2, 1)$

11. *Find all the eigen roots and the eigen vectors of the matrix*

$$A = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

>> $|A - \lambda I| = 0$ is the characteristic equation of A

i.e.,
$$\begin{vmatrix} (6-\lambda) & -2 & 2 \\ -2 & (3-\lambda) & -1 \\ 2 & -1 & (3-\lambda) \end{vmatrix} = 0$$

On expanding, we obtain

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$\lambda = 2$ is a root by inspection. Now by synthetic division,

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & +2 & -20 & +32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

Now by solving $\lambda^2 - 10\lambda + 16 = 0$ we obtain $(\lambda - 2)(\lambda - 8) = 0$

$\therefore \quad \lambda = 2, \ 8$

Thus **$\lambda = 2, \ 2, \ 8$ are the eigen values.**

We now form the equations,

$$\begin{aligned} (6-\lambda)x \quad -2y \quad\quad +2z \quad\quad &= 0 \\ -2x \quad +(3-\lambda)y \quad -1z \quad\quad &= 0 \\ 2x \quad -1y \quad\quad +(3-\lambda)z &= 0 \end{aligned} \qquad \ldots (1)$$

Case - (i) Let $\lambda = 2$ and the corresponding equations are,

$$4x - 2y + 2z = 0 \qquad \ldots \text{(i)}$$

$$-2x + y - z = 0 \qquad \ldots \text{(ii)}$$

$$2x - y + z = 0 \qquad \ldots \text{(iii)}$$

The above set of equations are all same as we have only one independent equation $2x - y + z = 0$ and hence we can choose two variables arbitrarily.

Let $\quad z = k_1$ and $y = k_2 \quad \therefore \quad x = (k_2 - k_1)/2$

$\therefore \quad X_1 = \left[(k_2 - k_1)/2, \ k_2, \ k_1 \right]'$ is the eigen vector corresponding to $\lambda = 2$ where $k_1, \ k_2$ are not simultaneously equal to zero.

Case - (ii) Let $\lambda = 8$ and we have from (1)

$$\left.\begin{aligned} -2x-2y+2z &= 0 \\ -2x-5y-1z &= 0 \\ 2x-1y-5z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{12} = \frac{-y}{6} = \frac{z}{6} \quad \text{or} \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore$ $X_2 = (2, -1, 1)$ **is the eigen vector corresponding to $\lambda = 8$.**

12. [illegible] vector

$$A = \begin{bmatrix} [illegible] \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

>> The charecteristic equation of A is $| A - \lambda I | = 0$.

ie., $$\begin{vmatrix} (-3-\lambda) & -7 & -5 \\ 2 & (4-\lambda) & 3 \\ 1 & 2 & (2-\lambda) \end{vmatrix} = 0$$

ie., $(-3-\lambda)\ [8-6\lambda+\lambda^2-6]+7\ [4-2\lambda-3]-5\ [4-4+\lambda] = 0$

ie., $(-3-\lambda)\ (\lambda^2-6\lambda+2)+7(1-2\lambda)-5(\lambda) = 0$

$-3\lambda^2+18\lambda-6-\lambda^3+6\lambda^2-2\lambda+7-14\lambda-5\lambda = 0$

ie., $-\lambda^3+3\lambda^2-3\lambda+1 = 0$

or $\lambda^3-3\lambda^2+3\lambda-1 = 0$

or $(\lambda-1)^3 = 0 \Rightarrow \lambda = 1, 1, 1$ **All the eigen values are equal.**

We now form the system of equations

$$\begin{aligned} (-3-\lambda)x - 7y \quad - 5z \quad &= 0 \\ 2x \quad + (4-\lambda)y + 3z \quad &= 0 \\ 1x \quad + 2y \quad + (2-\lambda)z &= 0 \end{aligned}$$

Putting $\lambda = 1$ we obtain,

$$\left.\begin{aligned} -4x-7y-5z &= 0 \\ 2x+3y+3z &= 0 \\ 1x+2y+1z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{-6} = \frac{-y}{-2} = \frac{z}{2} \quad \text{or} \quad \frac{x}{3} = \frac{y}{-1} = \frac{z}{-1}$$

Thus $X = (3, -1, -1)$, **is the eigen vector corresponding to the coincident eigen value $\lambda = 1$.**

13. [illegible] *the associated characteristic equation*

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

>> Sum of all the eigen values of the given matrix is equal to the 'trace' of the given matrix.

ie., $= 2+3+2 = 7$ (*Sum of the principal diagonal elements*)

Next, product of all the eigen values is equal to the determinant of the given matrix

ie., $\begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6-2) - 2(2-1) + 1(2-3) = 8-2-1 = 5$

Thus the sum and product of all the eigen values of the given matrix are respectively 7 and 5.

8.4 Similarity of matrices and Diagonalisation of matrices

Two square matrices A and B of the same order are said to be **similar** if there exists a non singular matrix P such that

$$B = P^{-1} A P$$

Here B is said to be similar to A.

Diagonalisation of a square matrix

Property *If A is a square matrix of order n having n linearly independent eigen vectors then there exists an n^{th} order square matrix P such that $P^{-1} A P$ is a diagonal matrix.*

We shall establish this result by considering a third order square matrix to make an important and interesting observation

Let A be a third order square matrix having eigen values $\lambda_1, \lambda_2, \lambda_3$ and the corresponding eigen vectors.

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Let the square matrix P be equal to $[\, X_1 \;\; X_2 \;\; X_3 \,]$.

ie., $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Now $AP = A[X_1 \ X_2 \ X_3] = [AX_1 \ AX_2 \ AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$

or $AP = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

ie., $AP = PD$ where D is the diagonal matrix represnted by

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Consider $AP = PD$

Pre multiplying by P^{-1} we have,

$$P^{-1}AP = P^{-1}PD = (P^{-1}P)D = ID = D$$

Thus $P^{-1}AP = D$

It is important to note that $P^{-1}AP$ is a diagonal matrix having the eigen values of A, $(\lambda_1, \lambda_2, \lambda_3)$ in its principal diagonal. We say that the matrix P diagonalizes A where P is constituted by the eigen vectors of A.

Note : (1) *The transformation of a square matrix A to $P^{-1}AP$ is known as* ***Similarity Transformation.***

(2) *The matrix P which diagonalizes A is called the modal matrix of A and the resulting diagonal matrix is called the* ***spectral matrix*** *of A.*

Computation of powers of a square matrix

Diagonalization of a square matrix A also helps us to find the powers of A: $A^2, A^3, A^4, \cdots$ etc.,

We have $D = P^{-1}AP$

$\therefore \quad D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}AIAP = P^{-1}A^2$

ie., $D^2 = P^{-1}A^2P$

Pre multiplying by P and post multiplying by P^{-1} we have,

$$P\,D^2\,P^{-1} = (PP^{-1})\,A^2\,(PP^{-1}) = I\,A^2\,I = A^2$$

ie., $\quad A^2 = P\,D^2\,P^{-1}$

Thus in general, $A^n = P\,D^n\,P^{-1}$, where

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

Working procedure for diagonalization of a square matrix A of order 3

- We find eigen values $\lambda_1, \lambda_2, \lambda_3$
- We find the eigen vectors X_1, X_2, X_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$
- We form the modal matrix $P = [\,X_1 \; X_2 \; X_3\,] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$
- We compute $P^{-1} = \dfrac{1}{|P|}\,(Adj\;P)$
- Finally we compute $P^{-1}AP$

The diagonalization of A is given by $D = P^{-1}AP$

where we obtain $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

WORKED PROBLEMS

14. *Reduce the matrix* $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ *to the diagonal form and hence find* A^4

>> The characteristic equation of A is $|A - \lambda I| = 0$.

ie., $\quad \begin{vmatrix} (-1-\lambda) & 3 \\ -2 & (4-\lambda) \end{vmatrix} = 0$

ie., $\quad (-1-\lambda)\,(4-\lambda) + 6 = 0$

ie., $\quad \lambda^2 - 3\lambda + 2 = 0$

or $\quad (\lambda - 1)\,(\lambda - 2) = 0 \quad \therefore \; \lambda = 1$ and 2 are the eigen values of A.

Now consider $[A - \lambda I]\ [X] = [0]$

ie., $\begin{bmatrix} (-1-\lambda) & 3 \\ -2 & (4-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ie., $(-1-\lambda)\, x + 3y = 0$

$-2x + (4-\lambda)\, y = 0$

Case - (i) : Let $\lambda = 1$

We get $-2x + 3y = 0$ or $2x = 3y$ or $\dfrac{x}{3} = \dfrac{y}{2}$

$X_1 = (3, 2)'$ is the eigen vector corresponding to $\lambda = 1$.

Case - (ii) : Let $\lambda = 2$

We get $-3x + 3y = 0$ or $x = y$ or $\dfrac{x}{1} = \dfrac{y}{1}$

$X_2 = (1, 1)'$ is the eigen vector corresponding to $\lambda = 2$.

Modal matrix $P = [X_1\ X_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

We have $|P| = 1$ and $P^{-1} = \dfrac{1}{|P|}(Adj P)$

$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

Now $P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Thus $P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **is the diagonal matrix.**

or $\quad P^{-1}AP = Diag(1, 2)$

Also we have $A^n = P D^n P^{-1}$

$\therefore \quad A^4 = P D^4 P^{-1}$ where $D^4 = \begin{bmatrix} 1^4 & 0 \\ 0 & 2^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

ie., $A^4 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -32 & 48 \end{bmatrix} = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$

Thus $A^4 = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$

15. *Diagonalize the matrix* $\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

>> Let $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$.

ie., $\begin{vmatrix} (-19-\lambda) & 7 \\ -42 & (16-\lambda) \end{vmatrix} = 0$

ie., $\lambda^2 + 3\lambda - 304 + 294 = 0$

ie., $\lambda^2 + 3\lambda - 10 = 0$

or $(\lambda - 2)(\lambda + 5) = 0$

ie., $\lambda = 2, -5$ are the eigen values of A.

Now consider, $[A - \lambda I][X] = [0]$

ie., $\begin{bmatrix} (19-\lambda) & 7 \\ -42 & (16-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ie., $(-19-\lambda)x + 7y = 0$

$-42x + (16-\lambda)y = 0$

Case-(i) : Let $\lambda = 2$

We get $-21x + 7y = 0$ and $-42x + 14y = 0$

ie., $y = 3x$ or $\dfrac{y}{3} = \dfrac{x}{1}$

$X_1 = (1, 3)'$ is the eigen vector corresponding to $\lambda = 2$.

Case-(ii) : Let $\lambda = -5$

We get $-14x + 7y = 0$ and $-42x + 21y = 0$

ie., $y = 2x$ or $\dfrac{y}{2} = \dfrac{x}{1}$

$\therefore$ $X_2 = (1, 2)'$ is the eigen vector corresponding to $\lambda = -5$.

Modal matrix $P = [x_1\, x_2] = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

We have $|P| = 2 - 3 = -1$ and $P^{-1} = \dfrac{1}{|P|}(\text{Adj } P)$

$$P^{-1} = -\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

Now $\quad P^{-1}AP = D = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -15 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

Thus $\quad P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ **is the diagonal matrix.**

or $\quad P^{-1}AP = Diag\,(2, -5)$

16 *Reduce the matrix* $A = \begin{bmatrix} & & \\ 7 & 2 & -5 \\ 10 & 4 & -6 \end{bmatrix}$ *into a diagonal matrix. Also find* A'

>> The characteristic equation of A is $|A - \lambda I| = 0$

ie., $\quad \begin{vmatrix} (11-\lambda) & -4 & -7 \\ 7 & (-2-\lambda) & -5 \\ 10 & -4 & (-6-\lambda) \end{vmatrix} = 0$

ie., $\quad (11-\lambda)[(-2-\lambda)(-6-\lambda)-20] + 4[7(-6-\lambda)+50]$
$$-7[-28-10(-2-\lambda)] = 0$$

ie., $\quad (11-\lambda)[\lambda^2+8\lambda-8]+4[8-7\lambda]-7[10\lambda-8] = 0$

ie., $\quad 11\lambda^2+88\lambda-88-\lambda^3-8\lambda^2+8\lambda+32-28\lambda-70\lambda+56 = 0$

ie., $\quad -\lambda^3+3\lambda^2-2\lambda = 0$ or $\lambda^3-3\lambda^2-2\lambda = 0$ or $\lambda(\lambda^2-3\lambda+2) = 0$

or $\quad \lambda(\lambda-1)(\lambda-2) = 0 \Rightarrow \lambda = 0, 1, 2$

Now consider $[A-\lambda I][X] = [0]$

ie., $\quad (11-\lambda)x - 4y - 7z = 0$

$\quad 7x + (-2-\lambda)y - 5z = 0$

$\quad 10x - 4y + (-6-\lambda)z = 0$

Case - (i) Let $\lambda = 0$ and the corresponding equations are

$$\left.\begin{aligned} 11x-4y-7z &= 0 \\ 7x-2y-5z &= 0 \\ 10x-4y-6z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{6} = \frac{-y}{-6} = \frac{z}{6} \quad \text{or} \quad \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$X_1 = (1, 1, 1)'$ is the eigen vector corresponding to $\lambda = 0$.

Case (ii) : Let $\lambda = 1$ and the corresponding equations are

$$\left.\begin{aligned} 10x - 4y - 7z &= 0 \\ 7x - 3y - 5z &= 0 \\ 10x - 4y - 7z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-2} \text{ or } \frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$$

$X_2 = (1, -1, 2)'$ is the eigen vector corresponding to $\lambda = 1$

Case (iii) : Let $\lambda = 2$ and the corresponding equations are

$$\left.\begin{aligned} 9x - 4y - 7z &= 0 \\ 7x - 4y - 5z &= 0 \\ 10x - 4y - 8z &= 0 \end{aligned}\right\} \Rightarrow \frac{x}{-8} = \frac{-y}{4} = \frac{z}{-8} \text{ or } \frac{x}{2} = \frac{y}{1} = \frac{z}{2}$$

$X_3 = (2, 1, 2)'$ is the eigen vector corresponding to $\lambda = 2$.

Hence the modal matrix $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

We have $\quad |P| = 1(-2-2) - 1(2-1) + 2(2+1) = 1$

$$Adj\ P = \begin{bmatrix} +(-2-2), & -(2-4), & +(1+2) \\ -(2-1), & +(2-2), & -(1-2) \\ +(2+1), & -(2-1), & +(-1-1) \end{bmatrix} = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(Adj\ P) = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$.

$$\text{Now,} \quad P^{-1}AP = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

Thus $\quad \mathbf{P^{-1}AP = D = Diag\,(0,\ 1,\ 2)}$

Further we have $A^n = P D^n P^{-1}$

$\therefore \quad A^5 = P D^5 P^{-1}$ and $D^5 = Diag\,(0^5, 1^5, 2^5) = Diag\,(0, 1, 32)$

$$\text{Hence, } A^5 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$\text{Thus } \quad A^5 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 96 & -32 & -64 \end{bmatrix} = \begin{bmatrix} \mathbf{191} & \mathbf{-64} & \mathbf{-127} \\ \mathbf{97} & \mathbf{-32} & \mathbf{-65} \\ \mathbf{190} & \mathbf{-64} & \mathbf{-126} \end{bmatrix}$$

17. *Diagonalize the matrix* $A = $ [illegible] $-6 \quad 7$ [illegible]

>> **Referring to problem-6**, we have the eigen values of A, $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 15$

and the corresponding eigen vectors are

$$X_1 = (1, 2, 2)', \quad X_2 = (2, 1, -2)', \quad X_3 = (2, -2, 1)'$$

$$\text{Hence the modal matrix } P = [X_1, X_2, X_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$|P| = 1(1-4) - 2(2+4) + 2(-4-2) = -27$$

$$Adj\ P = \begin{bmatrix} +(1-4), & -(2+4), & +(-4-2) \\ -(2+4), & +(1-4), & -(-2-4) \\ +(-4-2), & -(-2-4), & +(1-4) \end{bmatrix} = \begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(Adj\ P)$$

$$P^{-1} = \frac{1}{-27} \begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$.

$$\text{Now, } \quad P^{-1}AP = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix}$$

$$ie., \qquad P^{-1}AP = \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 135 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D$$

Thus $P^{-1}AP = D = Diag\ (0, 3, 15)$

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18 *Show that the matrix* $A = \begin{bmatrix} 7 & 2 & 0 \\ -2 & 6 & -2 \\ 0 & 2 & 5 \end{bmatrix}$ *is similar to its diagonal matrix. Find the associated diagonal matrix*

>> *Referring to Problem 7*, we have the eigen values of A, $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 9$ and the corresponding eigen vectors are,

$$X_1 = (1, 2, 2)', \ X_2 = (2, 1, -2)', \ X_3 = (2, -2, 1)'$$

Hence the modal matrix $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Remark. *The matrix P is same as in the previous problem and hence P^{-1} is also same as in the previous problem.*

Diagonalization of A is given by $P^{-1}AP$.

$$\text{Now,} \quad P^{-1}AP = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 12 & 18 \\ 6 & 6 & -18 \\ 6 & -12 & 9 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{9}\begin{bmatrix} 27 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} = D$$

Thus $P^{-1}AP = D = Diag\ (3, 6, 9)$

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19. *Find the modal matrix and the spectral matrix of A*

Hence compute the fourth power of the matrix A

>> **Referring to Problem-11,** we have the eigen values of A,

$$\lambda_1 = -2, \ \lambda_2 = 3, \ \lambda_3 = 6$$

and the corresponding eigen vectors are

$$X_1 = (1, 0, -1)', \ X_2 = (1, -1, 1)', \ X_3 = (1, 2, 1)'$$

Hence the **Modal matrix** $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

Now, $|P| = 1(-1-2) - 1(2+1) = -6$ *(Expanded by first column)*

$$Adj\ P = \begin{bmatrix} +(-1-2), & -(1-1), & +(2+1) \\ -(0+2), & +(1+1), & -(2-0) \\ +(0-1), & -(1+1), & +(-1-0) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}(Adj\ P) = \frac{-1}{6}\begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$

Now, $$P^{-1}AP = \frac{-1}{6}\begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \frac{-1}{6}\begin{bmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 3 & 6 \\ 0 & -3 & 12 \\ 2 & 3 & 6 \end{bmatrix}$$

$$P^{-1}AP = \frac{-1}{6}\begin{bmatrix} 12 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -36 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$$

Thus **Spectral matrix of** $A = D = Diag(-2, 3, 6)$

19. *Determine the eigen values and the corresp[illegible]n vectors of the matrix*

[illegible]

Also find the matrix P such that $P^{-1} AP$ is a diagonal matrix

>> **Referring to problem - 9** we have the eigen values of A $\lambda = -3, -3, 5$

The eigen vector corresponding to the coincident eigen value $\lambda = -3$ be denoted by $X_{1,2}$ and we have $X_{1,2} = (3k_1 - 2k_2, k_2, k_1)'$ where k_1, k_2 are arbitrary. We choose convenient values for k_1 and k_2 to obtain two distinct eigen vectors.

(i) Let $k_1 = 1,\ k_2 = 1 \quad \therefore \quad X_1 = (1, 1, 1)'$

(ii) Let $k_1 = 1,\ k_2 = 0 \quad \therefore \quad X_2 = (3, 0, 1)'$

Further we have obtained *(Problem-9)* the eigen vector corresponding to $\lambda = 5$ as $(1, 2, -1)'$

Denoting $X_3 = (1, 2, -1)'$, we have modal matrix

$$P = [X_1\, X_2\, X_3] = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|P| = 1(-2) - 3(-3) + 1(1) = 8$$

$$Adj\ P = \begin{bmatrix} +(0-2), & -(-3-1), & +(6-0) \\ -(-1-2), & +(-1-1), & -(2-1) \\ +(1-0), & -(1-3), & +(0-3) \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$

$$\text{Now,} \quad P^{-1}AP = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -3 & -9 & 5 \\ -3 & 0 & 10 \\ -3 & -3 & -5 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{8}\begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

Thus $\quad P^{-1}AP = D = Diag\,(-3, -3, 5)$

20 [illegible]

$$A = \begin{bmatrix} 6 & -2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

>> **Referring to problem-11**, we have the eigen values of A, $\lambda = 2, 2, 8$

The eigen vector corresponding to the coincident eigen values $\lambda = 2$ be denoted by $X_{1,2}$ and we have $X_{1,2} = \left[\dfrac{k_2 - k_1}{2}, k_2, k_1 \right]'$ where k_1, k_2 are arbitrary.

We choose convenient values for k_1 and k_2 to obtain two distinct eigen vectors which are orthogonal.

(i) Let $k_1 = 1, k_2 = 1 : X_1 = [0, 1, 1]'$

(ii) Suppose $X_2 = [a, b, c]'$ then we must have $X_1' \cdot X_2' = 0$

ie., $\quad 0 + b + c = 0$ or $b = -c$ or $\dfrac{b}{1} = \dfrac{c}{-1}$.

Since a is arbitrary, let us choose $a = 1$

$\therefore \quad X_2 = (1, 1, -1)'$ and we observe $X_1' \cdot X_2' = 0$

Further we have obtained the eigen vector corresponding to $\lambda = 8$ as $(2, -1, 1)'$

Denoting $X_3 = (2, -1, 1)'$ we also observe that $X_2' \cdot X_3' = 0$ and $X_3' \cdot X_1' = 0$

The modal matrix $P = [X_1 \, X_2 \, X_3] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$|P| = -1(1+1) + 2(-1-1) = -6$$

$$Adj\ P = \begin{bmatrix} +(1-1), & -(1+2), & +(-1-2) \\ -(1+1), & +(0-2), & -(0-2) \\ +(-1-1), & -(0-1), & +(0-1) \end{bmatrix}' = \begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}\,(Adj\,P) = \frac{-1}{6}\begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$

$$P^{-1}AP = \frac{-1}{6}\begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \frac{-1}{6}\begin{bmatrix} 0 & -3 & -3 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 16 \\ 2 & 2 & -8 \\ 2 & -2 & 8 \end{bmatrix}$$

$$= \frac{-1}{6}\begin{bmatrix} -12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -48 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = D$$

Thus $\quad P^{-1}AP = D = Diag\,(2,\,2,\,8)$

Remark :

1. *If the orthogonal eigen vectors $X_1,\ X_2,\ X_3$ are normalized then the associated modal matrix* $P = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$ *is an orthogonal matrix which also will give us*
$P^{-1}\,AP = P'\,AP = Diag\,(2,\,2,\,8)$.
2. *If orthogonal congruence was not specified we can arbitrarily choose k_1 and k_2 in $X_{1,\,2}$ to obtain two linearly independent eigen vectors (Similar to the previous problem) X_1 and X_2. Along with X_3 the associated modal matrix P will also give us*
$P^{-1}\,AP = Diag\,(2,\,2,\,8)$

---

21. *Show that the following matrix A is not diagonalizable*

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

>> The characteristic equation of A is $|\,A - \lambda I\,| = 0$

ie., $$\begin{vmatrix} (2-\lambda) & 1 & 0 \\ 0 & (2-\lambda) & 1 \\ 0 & 0 & (2-\lambda) \end{vmatrix} = 0$$

ie., $\quad (2-\lambda)^3 = 0 \Rightarrow \lambda = 2,\,2,\,2.$

The eigen vector corresponding to $\lambda = 2$ has to be obtained by solving the system of equations.

$$\begin{aligned} (2-2)x + 1y + 0z &= 0 \\ 0x + (2-2)y + 1z &= 0 \\ 0x + 0y + (2-2)z &= 0 \end{aligned}$$

ie., $\quad y = 0,\ z = 0\ ;\ x$ can be arbitrary.

$\therefore$ $x = k,\ y = 0,\ z = 0$ is the eigen vector corresponding to the coincident eigen value $\lambda = 2$.

It is evident that we cannot obtain three linearly independent eigen vectors

Thus we conclude that the matrix A is not diagonizable.

22. Find the symmetric matrix A having eigen valu [illegible] *vectors* $(1, 2, 2)',\ (2, 1, -2)',\ (2, -2, 1)$

>> Since the eigen vectors of a symmetric matrix are orthogonal we shall form the modal matrix with normalized eigen vectors.

Hence $$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

By data, the eigen values of A are $\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = 15$

$\therefore$ Diagonal matrix $D = Diag\ (0, 3, 15)$

We know that $D = P^{-1} AP$

Premultiplying by P and post multiplying by P^{-1} we have,

$$P D P^{-1} = P P^{-1} A P P^{-1} = I A I = A$$

But $P^{-1} = P'$ since P is orthogonal.

$$\therefore \quad A = P D P'$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & -6 \\ 30 & -30 & 15 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 72 & -54 & 18 \\ -54 & 63 & -36 \\ 18 & -36 & 27 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Thus the required symmetric matrix $$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Remark :

1. *Since we had normalized eigen vectors in P, P was an orthogonal matrix and hence we could use $P^{-1} = P'$ in the computation of $A = PDP^{-1}$. If P was formed with the actual eigen vectors, it would have been necessary to compute P^{-1} in the process of finding the symmetric matrix A.*
2. *Compare this problem with the earlier worked problem – 17*

22. If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ *is the given matrix and* $P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ *is the modal matrix find the value of* θ *which reduces A to the diagonal matrix*

>> We have, $P^{-1} AP = D$

Here $|P| = 1,\ Adj\ P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = P^{-1}$ since $|P| = 1$

$$P^{-1}AP = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a\cos\theta - b\sin\theta & a\sin\theta + b\cos\theta \\ b\cos\theta - c\sin\theta & b\sin\theta + c\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta(a\cos\theta - b\sin\theta) - \sin\theta(b\cos\theta - c\sin\theta) & \cos\theta(a\sin\theta + b\cos\theta) - \sin\theta(b\sin\theta + c\cos\theta) \\ \sin\theta(a\cos\theta - b\sin\theta) + \cos\theta(b\cos\theta - c\sin\theta) & \sin\theta(a\sin\theta + b\cos\theta) + \cos\theta(b\sin\theta + c\cos\theta) \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} (a\cos^2\theta - b\sin 2\theta + c\sin^2\theta) & (a - c)\sin\theta\cos\theta + b\cos 2\theta \\ (a - c)\sin\theta\cos\theta + b\cos 2\theta & a\sin^2\theta + b\sin 2\theta + c\cos^2\theta \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = D$$

This clearly implies that we must have,

$$(a - c)\sin\theta\cos\theta + b\cos 2\theta = 0$$

or $$(a - c)\frac{\sin 2\theta}{2} = -b\cos 2\theta$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{-2b}{a - c}$$

ie., $$\tan 2\theta = \frac{2b}{c - a} \Rightarrow 2\theta = \tan^{-1}\left(\frac{2b}{c - a}\right)$$

Thus the required $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{c - a}\right)$

Quadratic Forms

A homogeneous expression of second degree in any number of variables is called a *Quadratic Form*. (Q.F)

Examples :

1. $2x^2+3xy+4y^2$

2. $x_1^2+2x_2^2-3x_3^2+4x_1\,x_2-x_2\,x_3+6x_3\,x_1$

In general we have,

$$a_{11}\,x_1^2+2a_{12}\;x_1\,x_2+a_{22}\,x_2^2 \qquad \ldots\text{(i)}$$

$$a_{11}\,x_1^2+a_{22}\,x_2^2+a_{33}\,x_3^2+2\,a_{12}\,x_1\,x_2+2\,a_{23}\,x_2\,x_3+2\,a_{31}\,x_3\,x_1 \qquad \ldots\text{(ii)}$$

respectively representing quadratic forms in two and three variables.

It is possible to represent a quadratic form as a product of three matrices in the form $X'AX$ where X is the column matrix in the variables, A is a symmetric matrix and X' being the transpose of X is a row matrix.

With reference to (i) we have,

$$X'AX=[\,x_1\,x_2\,]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}$$

With reference to (ii) we have,

$$X'AX=[\,x_1\,x_2\,x_3\,]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix}$$

A is called the matrix of the quadratic form. We can easily write the matrix A of a given quadratic form and conversely given a symmetric matrix. We can write the associated quadratic form.

The symmetric matrix A associated with (i) and (ii) can be written as follows

(i) $$A=\begin{bmatrix} \textit{coeff. of } x_1^2 & \dfrac{1}{2}\textit{coeff. of } x_1\,x_2 \\ \dfrac{1}{2}\textit{coeff of } x_1\,x_2 & \textit{coeff. of } x_2^2\end{bmatrix}$$

(ii) $$A = \begin{bmatrix} \text{coeff. of } x_1^2 & \frac{1}{2}\text{ coeff. of } x_1 x_2 & \frac{1}{2}\text{ coeff. of } x_1 x_3 \\ \frac{1}{2}\text{ coeff. of } x_1 x_2 & \text{coeff. of } x_2^2 & \frac{1}{2}\text{ coeff. of } x_2 x_3 \\ \frac{1}{2}\text{ coeff of } x_1 x_3 & \frac{1}{2}\text{ coeff. of } x_2 x_3 & \text{coeff. of } x_3^2 \end{bmatrix}$$

Illustrative Examples

1. $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1 x_2 + 6x_1 x_3 + 8x_2 x_3$

>> $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix}$$

2. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1 x_2 - 2x_2 x_3 + 4x_3 x_1$

>> $$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

3. $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$

>> $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$$

4. $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

>> $$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

5. $16x_1^2 - x_2^2 + 3x_1 x_3 - 6x_2 x_3$

>> $$A = \begin{bmatrix} 16 & 0 & 3/2 \\ 0 & -1 & -3 \\ 3/2 & -3 & 0 \end{bmatrix}$$

6. $xy + yz + zx$

>> $$A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Further given a symmetric matrix, we can write the associated quadratic form easily.

Illustrative Examples

1. $ax^2 + 2hxy + by^2$

$$A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

2. $x^2 + y^2 + z^2 + 4xy + 6yz - 2zx$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

3. $x_1^2 + x_2^2 + x_3^2 + 4x_1 x_2 + 8x_1 x_3$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

[8.51] Reduction of a quadratic form into canonical form

Let $X' A X$ be the given quadratic form where A is a symmetric matrix.

Consider a linear transformation $X = PY$

Then
$$X' A X = (PY)' A (PY)$$
$$= (Y' P') A (PY)$$
$$= Y' (P' A P) Y$$

or
$$X' AX = Y' BY \quad \text{where} \quad B = P' AP$$

If $B = P' AP$ then B and A are ***congruent matrices*** Further the transformation $X = PY$ is called a ***congruent transformation***.

Canonical form : Rank, Index and Signature

If $B = P' AP$ is a diagonal matrix, then the transformed quadratic form $Y' BY$ is a sum of square terms known as ***canonical form***. $X' AX$ is transformed into the form $d_1 y_1^2 + d_2 y_2^2 + \cdots + d_n y_n^2$ being the canonical form.

The *rank (r) of B or A is called the* ***rank of the quadratic form.***

The number of positive terms in the canonical form of a quadratic form is known as the ***Index (p) of the quadratic form***.

The difference between the number of positive terms and negative terms in the canonical form is known as the ***signature*** of the quadratic form.

Note : *$B = Diag\ (d_1, d_2, \ldots d_n)$ can further be reduced to*
$B = Diag\ (\pm 1, \pm 1, \pm 1, \quad \pm 1)$ and the associated canonical form will be
$\pm y_1^2 \pm y_2^2 \pm \ldots \pm y_n^2$.

8.52 Nature of quadratic form

If r is the rank and p is the index of the quadratic form in n variables the nature of the quadratic form is identified as presented in the following table

	Condition	Nature of Q.F	Canonical form	Remark on canonical form
1.	$r = n, p = n$	Positive definite	$y_1^2 + y_2^2 + \quad + y_n^2$	Only positive terms (n terms)
2.	$r = n, p = 0$	Negative definite	$-y_1^2 - y_2^2 \ldots - y_n^2$	Only negative terms (n terms)
3.	$r = p, p < n$	Positive semi definite	$y_1^2 + y_2^2 + \quad + y_r^2$	Only positive terms (r terms)
4.	$r < n, p = 0$	Negative semi-defnite	$-y_1^2 \quad y_2^2 \quad\quad y_r^2$	Only negative terms (r terms)

In all other cases the quadratic form is said to be **indefinite**. Indefinite quadratic form will contain both positive and negative terms in the canonical form.

Note : ***Orthogonal Transformation***

Suppose $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A having corresponding orthogonal eigen vectors X_1, X_2, X_3 in the normalized form, the associated modal matrix P will be an orthogonal matrix. ($P^{-1} = P'$ is this case)

We have in this case,

$$P^{-1}\ AP = P'\ AP = D = Diag\ (\lambda_1, \lambda_2, \lambda_3)$$

The associatead canonical form will be $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

Accordingly the nature of the quadratic form is presented in the following table.

	Nature of quadratic form	Nature of eigen values
1	Positive definite	Positive eigen values
2.	Negative definite	Negative eigen values
3	Positive semi definite	Positive eigen values at least one is zero.
4.	Negative semi definite	Negative eigen values atleast one is zero

In the case of indefinite quadratic form there will be positive as well as negative eigen values.

[illegible] procedure [illegible] to reduce [illegible] quadratic form to sum of squares (canonical form)

Case-(i) ***By canonical transformation***

- We write the matrix A of the Q.F.
- We write $A = I\,A\,I$
- We perform elementary transformation to reduce A to the diagonal form
- The elementary row transformations are also performed on the premultiplied I where as the column transformations are performed on the post multiplied I
- We obtain $D = P'AP$
- If $D = Diag\,(d_1, d_2, d_3)$ in respect of a third order square matrix A, the canonical form of the given Q.F is $d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2$
- $X = PY$ where $Y = [\,y_1\, y_2\, y_3\,]$ and $X = [\,x_1\, x_2\, x_3\,]'$ will give us the congruent linear transformation
- If required we can also obtain the canonical form as $\pm y_1^2 \pm y_2^2 \pm y_3^2$

Case-(ii) ***By orthogonal transformation***

- We write the matrix A of the Q.F
- We obtain the eigen values $\lambda_1, \lambda_2, \lambda_3$ and the corresponding orthogonal eigen vectors X_1, X_2, X_3 of the third order square matrix A.
- We normalize the orthogonal vectors X_1, X_2, X_3 and write the associated orthogonal modal matrix P.
- Since $P^{-1} = P'$ in this case we have $P'AP = Diag\,(\lambda_1, \lambda_2, \lambda_3)$
- The associated canonical form is $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$
- $X = PY$ where $Y = [\,y_1, y_2, y_3\,]'$ and $X = [\,x_1, x_2, x_3\,]'$ will give us the orthogonal linear transformation.

WORKED PROBLEMS

>> The symmetric matrix A of the given Q.F is $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(a) *By congruent transformation*

Let $A = I\,A\,I$

ie.,
$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -1/2 \cdot R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -1/2 \cdot C_1 + C_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ie., $D = P'\,AP$

where $P = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The canonical form is $Y'\,DY$ where $Y = [\,y_1\; y_2\; y_3\,]'$

ie.,
$$Y'\,DY = [\,y_1\; y_2\; y_3\,] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Thus $2\,y_1^2 + 2\,y_2^2 + (3/2)\,y_3^2$ *is the canonical form* of the given quadratic form

The associatead congruent linear transformation is $X = PY$

ie., $x_1 = y_1 - (1/2)\,y_2,\; x_2 = y_2,\; x_3 = y_3$

(b) *By orthogonal transformation*

We have to first compute the eigen values and the corresponding eigen vector of A

Referring to problem-8, we have obtained

$$\lambda_1 = 1,\; \lambda_2 = 2,\; \lambda_3 = 3$$

$$X_1 = (-1, 0, 1)',\; X_2 = (0, 1, 0)',\; X_3 = (1, 0, 1)'$$

We normalize these vectors and write the associated modal matrix P

ie., $P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

The orthogonal transformation $X = PY$ transforms the quadrataic form $X'AX$ into $Y'DY$ where $D = P^{-1}\,AP = P'AP$ is the diagonal matrix given by

$$D = Diag\,(\lambda_1, \lambda_2, \lambda_3) = Diag\,(1,\ 2,\ 3)$$

Thus $y_1^2 + 2y_2^2 + 3y_3^2$ is the canonical form of the given Q.F.

The associated orthogonal linear transformation $X = PY$ is given by

$$x_1 = (-1/\sqrt{2})\,y_1 + (1/\sqrt{2})\,y_3,\ \ x_2 = y_2,\ \ x_3 = (1/\sqrt{2})\,y_1 + (1/\sqrt{2})\,y_3$$

Remark :

1. *Rank of the Q.F = Rank of A = 3*

   Index of the Q F = No. of positive terms in the canonicl form = 3

   Signature of the Q.F = Difference between the number of positive and negative terms
   $= 3 - 0 = 3$

   Nature of the Q.F is positive definite.

2. *If orthogonal transformation is not specified, we always adopt congrue transformation to reduce the given Q.F into sum of squares.*

25. *Find the transf*
squares and find th

$4x^2 + 3y^2 + z^2 - 8xy - 6yz + 4$

>> The symmetric matrix A of the given Q F is $A = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix}$

Let $A = I\,A\,I$

ie., $\begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_1 + R_2,\ \ R_3 \to -1/2 \cdot R_1 + R_3$

$$\begin{bmatrix} 4 & -4 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_1 + C_2,\ \ C_3 \to -1/2 \cdot C_1 + C_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\,(4, -1, 1) = P'AP$, where $P = \begin{bmatrix} 1 & 1 & -3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$X = PY$ is the congruent transformation which has transformed the given quadratic form into sum of squares given by,

$$4u^2 - v^2 + w^2$$

Where $Y = [\,u\ v\ w\,]'$ and $X = [\,x\ y\ z\,]'$

The **congruent transformation** is given by $x = u + v - (3/2)\,w,\ y = v - w,\ z = w$

Remark : ***Nature of the quadratic form***

Rank $= 3$, *Index* $= 2$, *Signature* $= 2 - 1 = 1$

Q.F. is indefinite since the canonical form has both positive and negative terms.

---

26. *Reduce the quadratic form* $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1 x_2 + 8x_2 x_3$ *into sum of squares.*

>> The symmetric matrix A of the given Q.F. is

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & 4 \\ 0 & 4 & -7 \end{bmatrix}$$

Let $A = I\,A\,I$

i.e., $$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & 4 \\ 0 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 2R_1 + R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \rightarrow 2C_1 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow 2R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \rightarrow 2C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = \text{Diag}(1, -2, 1) = P'AP$, where $P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$X = PY$ is the congruent transformation where $X = [x_1 \ x_2 \ x_3]'$ and $Y = [y_1 \ y_2 \ y_3]'$

ie., $x_1 = y_1 + 2y_2 + 4y_3$, $x_2 = y_2 + 2y_3$, $x_3 = y_3$

Canonical form is $y_1^2 - 2y_2^2 + y_3^2$

Remark : ***Nature of the quadratic form***

Rank = 3, *Index* = 2, *Signature* = 1, *Indefinite form*

27. [illegible] *transformation. Indicate the transformation and also find the nature* [illegible]

[illegible] The symmetric matrix A of the given form is $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 2 \\ 3 & -2 & 3 \end{bmatrix}$

Let $\quad A = I\,A\,I$

ie., $\quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 \to -3R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -3C_1 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Further we need to make the diagonal elements 1 numerically Hence we perform the transformations

$$\frac{1}{\sqrt{2}}\,R_2,\ \frac{1}{\sqrt{2}}\,C_2 \ \text{ and } \ \frac{1}{2}R_3,\ \frac{1}{2}\,C_3$$

ie., $\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ -3/2 & -1/2 & 1/2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$

We have $D = Diag\,(1, -1, -1) = P'\,AP$

The canonical form is $y_1^2 - y_2^2 - y_3^2$ under the congruent transformation $X = PY$,

where $X = [\,x\ y\ z\,]'$, $Y = [\,y_1\ y_2\ y_3\,]'$ and $P = \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$

The congruent transformation is $x = y_1 - (3/2)\,y_2$, $y = (1/\sqrt{2})\,y_2 - (1/2)\,y_3$, $z = (1/2)\,y_3$

The **quadratic form is indefinite** as the canonical form contains both positive and negative terms.

---

28 *Find the rank, index, signature of the following quadratic form*

$$2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6zx$$

\>> The symmetric matrix A of the given Q.F is

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix}$$

Let $\quad A = I\,A\,I$

ie.,

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 1/2 \cdot R_1 + R_2,\ R_3 \to -3/2 \cdot R_1 + R_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & -5/2 & -5/2 \\ 0 & -5/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 1/2 \cdot C_1 + C_2;\ C_3 \to -3/2 \cdot C_1 + C_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & -5/2 \\ 0 & -5/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = \mathrm{Diag}(2, -5/2, 0) = P'AP$

The canonical form is $2y_1^2 - (5/2)\,y_2^2$ under the congruent transformation $X = PY$

where $X = [\,x\ y\ z\,]'$, $Y = [\,y_1\ y_2\ y_3\,]'$ and $P = \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

The congruent transformation is

$$x = y_1 + (1/2)\,y_2 - 2y_3\,,\quad y = y_2 - y_3\,,\quad z = y_3$$

The quadratic form has,

Rank = 2, Index = 1, Signature = 0 and the nature is indefinite.

---

29. *Reduce the following symmetric matrix to diagonal form by congruent transformation and interpret the result in terms of quadratic form*

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

>> Let $A = I\,A\,I$

ie., $$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 1/3 \cdot R_1 + R_2\,,\ R_3 \to -1/3 \cdot R_1 + R_3$

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 1/3 \cdot C_1 + C_2\,,\ C_3 \to -1/3 \cdot C_1 + C_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/7\ R_2 + R_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 1/7 \cdot C_2 + C_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\,(6\,,\ 7/3\,,\ 16/7) = P'AP$, where $P = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$

Interpretation

The quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$ is reduced to the canonical form :

$$6y_1^2 + (7/3)\,y_2^2 + (16/7)\,y_3^2$$

under the transformation $X = PY$ given by :

$$x_1 = y_1 + (1/3)\,y_2 - (2/7)\,y_3\,,\quad x_2 = y_2 + (1/7)\,y_3\,,\quad x_3 = y_3$$

Further the quadratic form has,

Rank = 3, Index = 3, Signature = 3 and is positive definite in nature

Remark : *Canonical form of this quadratic form by orthogonal transformation*

Referring to problem 20 and the Remark made in that problem we have

$P'AP = Diag\,(2\,,\ 2\,,\ 8)$ where P is the orthogonal matrix given by

$$P = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

The canonical form by orthogonal transformation $X = PY$ is $2y_1^2 + 2y_2^2 + 8y_3^2$
The rank, index, signature and the nature is the same as stated earlier

---

30. [illegible] $3y^2$ [illegible] its nature

>> The symmetric matrix A of the given Q.F is

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Let $\quad A = I\,A\,I$

i.e.,
$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -1/3 \cdot R_1 + R_2,\ R_3 \to -1/3 \cdot R_1 + R_3$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & -4/3 \\ 0 & -4/3 & 8/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to -1/3\ C_1 + C_2,\ C_3 \to -1/3 \cdot C_1 + C_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & -4/3 \\ 0 & -4/3 & 8/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/2 \cdot R_2 + R_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & -4/3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 1/2 \cdot C_2 + C_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\,(3,\ 8/3,\ 2) = P'AP$ where $P = \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

The canonical form is $3y_1^2 + (8/3)\,y_2^2 + 2y_3^2$ under the congruent transformation

$X = PY$ where $X = [\,x\ y\ z\,]'$ and $y = [\,y_1\ y_2\ y_3\,]'$.

The quadratic form is positive definite.

31. *Show that the following quadratic form is positive semidefinite. Also find a non zero set of values of x_1, x_2, x_3 which makes the form zero*

$$5x_1^2 + 26x_2^2 + 10x_3^2 + 6x_1 x_2 + 14x_1 x_3 + 4x_2 x_3$$

>> The symmetric matrix A of the Q.F is

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Let $A = I\,A\,I$

ie., $$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -3/5 \cdot R_1 + R_2$, $R_3 \to -7/5 \cdot R_1 + R_3$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121/5 & -11/5 \\ 0 & -11/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -7/5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to -3/5 \cdot C_1 + C_2$, $C_3 \to -7/5 \cdot C_1 + C_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & -11/5 \\ 0 & -11/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -7/5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -7/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/11 \cdot R_2 + R_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & -11/5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -16/11 & 1/11 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -7/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 1/11\ C_2 + C_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -16/11 & 1/11 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -16/11 \\ 0 & 1 & 1/11 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $D = Diag\ (5,\ 121/5,\ 0) = P'AP$ where

$$P = \begin{bmatrix} 1 & -3/5 & -16/11 \\ 0 & 1 & 1/11 \\ 0 & 0 & 1 \end{bmatrix}$$

The canonical form is $5\,y_1^2 + (121/5)\ y_2^2$

The quadratic form has,

Rank = 2, Index = 2

Since the Rank = Index = 2 < 3 the quadratic form is positve semidefinite.

The congruent transformation $X = PY$ is given by

$$x_1 = y_1 - (3/5)\,y_2 - (16/11)\ y_3,\quad x_2 = y_2 + (1/11)\ y_3,\quad x_3 = y_3$$

Further $y_1 = 0$ and $y_2 = 0$ will reduce the quadratic form to zero. y_3 can be arbitrary, $y_3 = 1$ (say)

Hence $x_1 = -16/11$, $x_2 = 1/11$, $x_3 = 1$ is a set of non zero values that makes the quadratic form zero.

ical form by orth
nature of the quadratic form

\>> The symmetric matrix of the Q.F is

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Referring to problem - 17 we have,

$\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = 15$ and the corresponding eigen vectors are

$X_1 = [1, 2, 2]',\ X_2 = [2, 1, -2]',\ X_3 = [2, -2, 1]'$

The modal matrix P consisting normalized eigen vectors is

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$P^{-1} = P'$ since P is an orthogonal matrix.

We have $D = Diag\,(0,\ 3,\ 15) = P'AP$

The canonical form is $3y_2^2 + 15y_3^2$

The quadratic form has,

Rank = 2, **Index** = 2, **Signature** = 2 **and it is positive semidefinite.**

Further the orthogonal transformation $X = PY$ is given by

$$x = \frac{1}{3}(y_1 + 2y_2 + y_3),\ y = \frac{1}{3}(2y_1 + y_2 - 2y_3),\ z = \frac{1}{3}(2y_1 - 2y_2 + y_3)$$

>> The symmetric matrix A of the Q.F is

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Referring to problem - 19 we have

$\lambda_1 = -2,\ \lambda_2 = 3,\ \lambda_3 = 6$ and the corresponding eigen vectors

$X_1 = [1,\ 0,\ -1]',\ X_2 = [1,\ -1,\ 1]',\ X_3 = [1,\ 2,\ 1]'$

The modal matrix P consisting normalized eigen vectors is

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$P^{-1} = P'$ since P is an orthogonal matrix.

We have $D = Diag\ (-2,\ 3,\ 6) = P'AP$

The canonical form is $-2\,y_1^2 + 3\,y_2^2 + 6\,y_3^2$

The orthogonal transformation $X = PY$ is given by

$$x_1 = \frac{1}{\sqrt{2}}\,y_1 + \frac{1}{\sqrt{3}}\,y_2 + \frac{1}{\sqrt{6}}\,y_3,\ x_2 = \frac{-1}{\sqrt{3}}\,y_2 + \frac{2}{\sqrt{6}}\,y_3,\ x_3 = \frac{1}{\sqrt{2}}\,y_1 + \frac{1}{\sqrt{3}}\,y_2 + \frac{1}{\sqrt{6}}\,y_3$$

>> $X = [x\ y\ z\ w]'$, $X'AX$ is the qudratic form in four variables where the symmetric matrix A is given by

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & 3 & 6 & -4 \\ 3 & 6 & 8 & -6 \\ -2 & -4 & 6 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3/2 & -2 \\ -1 & -3 & -5/2 & 3 \\ 3/2 & -5/2 & 4 & 1/2 \\ -2 & 3 & 1/2 & 1 \end{bmatrix}$$

>> Let $X = [x_1 \ x_2 \ x_3 \ x_4]'$. The Q.F $X' A X$ is given by

$$2x_1^2 - 3x_2^2 + 4x_3^2 + x_4^2 - 2x_1x_2 + 3x_1x_3 - 4x_1x_4 - 5x_2x_3 + 6x_2x_4 + x_3x_4$$

EXERCISES

1. Show that the transformation $y_1 = 2x_1 + x_2 + x_3$, $y_2 = x_1 + x_2 + 2x_3$, $y_3 = x_1 - 2x_3$ is regular. Find the inverse transformation.

Find all the eigen values and the corresponding eigen vectors for the following matrices.

2. $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$

5. Find a matrix P which transforms the following matrix A to diagonal form.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \quad \text{Hence find } A^4$$

6. Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

7. Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is similar to its diagonal matrix. Also find transforming matrix and diagonal matrix.

8. Reduce the following quadratic form into canonical form by congruent transformation and give the corresponding linear transformation.

$$10x_1^2 + x_2^2 + x_3^2 - 6x_1x_2 - 2x_2x_3 + x_3x_1$$

9. Reduce the following quadratic form into sum of squares by an orthogonal transformation. Give the matrix and nature of the form.

$$3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$$

10. Reduce to sum of squares the quadratic form :

$$x^2 + 2y^2 - 7z^2 - 4xy + 8yz$$

Find the rank, index, signature and the nature of the form

ANSWERS

1. $x_1 = 2y_1 - 2y_2 - y_3, \ x_2 = -4y_1 + 5y_2 + 3y_3, \ x_3 = y_1 - y_2 - y_3$

2. $\lambda = 0, -2, 1; \ (10, 3, -11), \ (4, 3, -7), \ (1, 0, 1)$

3. $\lambda = 1, 1, 5; \ [-(2k_1 + k_2), k_1, k_2], \ (1, 1, 1)$

4. $\lambda = 1, 1, 1; \ (k_1, 3k_1, k_2)$

5. $P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}; \ P^{-1}AP = Diag\,(1, 2, 3)$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

6. $P^{-1}AP = D = Diag\,(1, 2, 3)$ where, $P = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

7. $P^{-1}AP = D = Diag\,(1, 2, 3)$ where, $P = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

8. $10y_1^2 + \frac{1}{10}y_2; \ x_1 = y_1 + \frac{3}{10}y_2, \ x_2 = y_2 + y_3, \ x_3 = y_3$

9. $y_1^2 + 4y_2^2 + 4y_3^2; \ \begin{bmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$, positive definite.

10. $y_1^2 - 2y_2^2 + 9y_3^2$. Rank = 3, Index = 2, Signature = 1, Indefinte form.

BEATING THE MEMORY

[*Formulae, Properties and Results to be remembered from all the units at a glance*]

Unit I — DIFFERENTIAL CALCULUS I

Table of n^{th} derivatives of standard functions

	$y = f(x)$	$y_n = D^n y$
F_1	e^{ax}	$a^n e^{ax}$
F_2	a^{mx}	$(m \log a)^n a^{mx}$
F_3	$(ax+b)^m, m > n$	$m(m-1)(m-2)\ldots[m-(n-1)]a^n(ax+b)^{m-n}$
F_4	$\dfrac{1}{ax+b}$	$\dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$
F_5	$\log(ax+b)$	$\dfrac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$
F_6	$\sin(ax+b)$	$a^n \sin\left(\dfrac{n\pi}{2} + ax + b\right)$
F_7	$\cos(ax+b)$	$a^n \cos\left(\dfrac{n\pi}{2} + ax + b\right)$
F_8	$e^{ax}\sin(bx+c)$	$(\sqrt{a^2+b^2})^n e^{ax}\sin[n\tan^{-1}(b/a) + bx + c]$
F_9	$e^{ax}\cos(bx+c)$	$(\sqrt{a^2+b^2})^n e^{ax}\cos[n\tan^{-1}(b/a) + bx + c]$

Remark :

Observe similarities in the pair of formulae F_4 & F_5 ; F_6 & F_7 ; F_8 & F_9 as it would help to remember the formulae easily.

Leibnitz theorem for the n^{th} derivative of a product

$$D^n(uv) \text{ or } (uv)_n = uv_n + nu_1 v_{n-1} + \frac{n(n-1)}{1.2} u_2 v_{n-2} + \cdots + u_n v$$

Rolle's theorem

If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists atleast one point c in (a, b) such that $f'(c) = 0$

Lagrange's mean value theorem

If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) then there exists atleast one point c in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Cauchy's mean value theorem

If $f(x)$ and $g(x)$ are two continuous functions in $[a, b]$, differentiable in (a, b) with $g'(x) \neq 0$ for all x in (a, b) then there exists atleast one point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Expansion of a function $y(x)$

- *Taylor's expansion : (about $x = a$)*

$$y(x) = y(a) + (x-a)\,y_1(a) + \frac{(x-a)^2}{2!}\,y_2(a) + \frac{(x-a)^3}{3!}\,y_3(a) + \cdots$$

- *Maclaurin's expansion* (about $x = 0$)

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + \cdots$$

Unit - II DIFFERENTIAL CALCULUS

Indeterminate forms

- *L Hospital's rule (for 0/0 and ∞/∞ forms)*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} \text{ etc}$$

Polar curves

- *Angle (ϕ) between the radius vector and the tangent*

$$\tan\phi = r\frac{d\theta}{dr} \quad \text{or} \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

- *Length of the perpendicular (p) from the pole to the tangent*

$$p = r\sin\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

- *Angle of intersection of two polar curves is given by* $|\phi_1 - \phi_2|$

If $|\phi_1 - \phi_2| = \pi/2$ or $\tan\phi_1 \cdot \tan\phi_2 = -1$

then the curves intersect each other orthogonally or at right angles

➢ *Pedal equation (p-r equation) of a polar curve*

If θ is eliminated from the given equation $r = f(\theta)$ and $p = r\sin\phi$, where ϕ is usually a function of θ, the resulting equation in p and r is the pedal equation of the polar curve.

Radius of curvature

➢ *Curvature* : $K = \dfrac{d\psi}{ds}$, *Radius of curvature* $\rho = \dfrac{ds}{d\psi}$

➢ *Cartesian curve* :

$$[\, y = y(x)\,],\ \rho = \frac{(1+y_1^2)^{3/2}}{y_2};\quad [\, x = x(y)\,],\ \rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

➢ *Parametric curve* : $[\, x = x(t),\ y = y(t)\,]$, $\rho = \dfrac{[(\dot{x})^2+(\dot{y})^2]^{3/2}}{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}$

➢ *Polar curve* : $[\, r = f(\theta)\,]$, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

➢ *Pedal curve* : $[\, r = f(p)\,]$, $\rho = r\dfrac{dr}{dp}$

UNIT - III DIFFERENTIAL CALC

Partial Differentiation

Partial derivatives of u (x , y)

First order partial derivatives

$u_x = \dfrac{\partial u}{\partial x}$ $u_y = \dfrac{\partial u}{\partial y}$

Second order partial derivatives

$u_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}$ $u_{yy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2}$

$u_{yx} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial y\,\partial x}$ ⟵ **Mixed partial derivatives** ⟶ $u_{xy} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial x\,\partial y}$

⟶ and so-on ⟵

Further, $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$ or $u_{yx} = u_{xy}$

Differentiation of composite functions

If $u = u(x, y)$ where $x = x(t)$ and $y = y(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \textit{(Total derivative)}$$

If $z = z(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$ then

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\} \quad \textit{(Chain rule)}$$

Jacobians

If u, v, w are all functions of x, y, z then the *jacobian* (J) is given by

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

Taylor's series expansion of $f(x, y)$ about (a, b) and about $(0, 0)$

$$f(x, y) = f(a, b) + \frac{1}{1!}\left\{(x-a)f_x(a, b) + (y-b)f_y(a, b)\right\}$$
$$+ \frac{1}{2!}\left\{(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\right\} + \cdots$$

In particular if $(a, b) = (0, 0)$, the series is called as **Taylor's series about the origin or Maclaurin's series** given by

$$f(x, y) = f(0, 0) + \frac{1}{1!}\left\{x f_x(0, 0) + y f_y(0, 0)\right\}$$
$$+ \frac{1}{2!}\left\{x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right\} + \ldots$$

Maxima and Minima of $f(x, y)$

Working procedure for finding extreme values of $f(x, y)$

(i) We have to first find the stationary points (x, y) such that $f_x = 0$ and $f_y = 0$

(ii) We then find the second order partial derivatives :
$A = f_{xx}, \ B = f_{xy}, \ C = f_{yy}$

We evaluate these at all the stationary points and also compute the corresponding value of $AC - B^2$

(iii) (a) A stationary point (x_0, y_0) is a maximum point if $AC - B^2 > 0$ & $A < 0$; $f(x_0, y_0)$ is a maximum value.

(b) A stationary point (x_1, y_1) is a minimum point if $AC - B^2 > 0$ & $A > 0$; $f(x_1, y_1)$ is a minimum value.

Note : *We can overlook the cases of* $AC - B^2 < 0$, $AC - B^2 = 0$, $A = 0$

UNIT - IV VECTOR CALCULUS

Vector Differentiation

Vector differential operator 'Nabla' (∇)

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

If $\phi(x, y, z)$ is a scalar point function and $\vec{A}(x, y, z)$ is a vector point function, then

$\nabla\phi$	= grad ϕ	= Gradient of ϕ
$\nabla \cdot \vec{A}$	= div $\vec{A}$	= Divergence of $\vec{A}$
$\nabla \times \vec{A}$	= curl $\vec{A}$	= Curl of $\vec{A}$
$\nabla \cdot \nabla \phi$	= div (grad ϕ)	= Laplacian of $\phi = \nabla^2 \phi$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the Laplacian operator

Geometrical meaning of $\nabla\phi$

If $\phi(x, y, z) = c$ be the equation of a surface, then $\nabla\phi$ is a *vector normal* to the surface

➤ $\nabla\phi \cdot \hat{n}$ is the *directional derivative* of ϕ along a given direction $\vec{A}$ where

$\vec{A} / |\vec{A}| = \hat{n}$

- The *angle between two surfaces* is equal to the angle between their normals and if this angle is equal to 90° then the surfaces are said to be orthogonal to each other
- A vector $\vec{A}$ is said to be **solenoidal** if $\text{div}\,\vec{A} = 0$ and **irrotational** (*conservative*) if $\text{curl}\,\vec{A} = \vec{0}$
- If $\vec{A}$ is irrotational there always exists a scalar function ϕ such that $\nabla\phi = \vec{A}$ and ϕ is called the *scalar potential* of $\vec{A}$

List of vector identities

1. $\text{curl}\,(\text{grad}\,\phi) = \vec{0}$
2. $\text{div}\,(\text{curl}\,\vec{A}) = 0$
3. $\text{curl}\,(\text{curl}\,\vec{A}) = \text{grad}\,(\text{div}\,\vec{A}) - \nabla^2 \vec{A}$
4. $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$
5. $\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$
6. $\text{div}\,(\vec{A} \times \vec{B}) = \vec{B} \cdot \text{curl}\,\vec{A} - \vec{A} \cdot \text{curl}\,\vec{B}$

Orthogonal Curvilinear Coordinates (O.C.C)

Curvilinear coordinates : (u_1, u_2, u_3) and $\vec{r} = \vec{r}(u_1, u_2, u_3)$

Scale factors and unit vectors

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|,$$

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1}, \quad \hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2}, \quad \hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3}$$

Orthogonal system	Coordinates (u_1, u_2, u_3)	Transformation	Scale factors and unit vectors
Cylindrical system	(ρ, ϕ, z) [cylindrical polar coordinates]	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$h_1 = 1, \hat{e}_\rho$ $h_2 = \rho, \hat{e}_\phi$ $h_3 = 1, \hat{e}_z$
Spherical system	(r, θ, ϕ) [Spherical polar coordinates]	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$h_1 = 1, \hat{e}_r$ $h_2 = r ; \hat{e}_\theta$ $h_3 = r\sin\theta ; \hat{e}_\phi$
Cartesian system	(x, y, z) [Cartesian coordinates]	$x = x$ $y = y$ $z = z$	$h_1 = 1 ; \hat{i}$ $h_2 = 1 ; \hat{j}$ $h_3 = 1 ; \hat{k}$

Expression for the Arc length and the Volume element in O.C.C

$$ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2 \quad \text{[Arc length]}$$

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3 \qquad \text{[Volume element]}$$

Expression for Gradient, Divergence, Curl and Laplacian in O.C.C

Let $\psi = \psi_1(u_1, u_2, u_3)$ be a scalar point function and

$\vec{A} = \vec{A}(u_1, u_2, u_3) = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ be a vector point function.

$$\text{Grad } \psi = \nabla \psi = \sum \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1$$

$$\text{Div } \vec{A} = \nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \sum \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

$$\text{Laplacian of } \psi = \nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \sum \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right)$$

UNIT - V INTEGRAL CALCULUS

Differentiation under the Integral sign

- **Leibnitz rule**

If $\phi(\alpha) = \int_a^b f(x, \alpha)\, dx$ where a and b are constants, then

$$\phi'(\alpha) = \frac{d\phi}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha} [f(x, \alpha)]\, dx$$

Reduction formulae

➤ $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \times k$

where $k = 1$ when n is odd and $k = \pi/2$ when n is even.

➤ $\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)\ldots][(n-1)(n-3)\ldots]}{(m+n)(m+n-2)(m+n-4)\ldots} \times k$

where $k = \pi/2$ only when m and n are even integers.

Applications of Integral calculus

Derivative of Arc Length

(i) $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ (ii) $\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$

(iii) $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ (iv) $\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$

(v) $\frac{ds}{dr} = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}$

Integration with respect to the corresponding independent variable will give s

Applications formulat at a glance

	Cartesian curve	Parametric curve	Polar curve
Area (A)	$\int_a^b y \, dx$ or $\int_c^d x \, dy$	$\int_{t_1}^{t_2} y \frac{dx}{dt} dt$ or $\int_{t_1}^{t_2} x \frac{dy}{dt} dt$	$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$
Length (s)	$\int_a^b \frac{ds}{dx} dx$ or $\int_c^d \frac{ds}{dy} dy$	$\int_{t_1}^{t_2} \frac{ds}{dt} dt$	$\int_{\theta_1}^{\theta_2} \frac{ds}{d\theta} d\theta$ or $\int_{r_1}^{r_2} \frac{ds}{dr} dr$

	Cartesian curve	Parametric curve	Polar curve
Surface area of revolution (S)	$2\pi \int_a^b y \frac{ds}{dx}\, dx$ (about the x-axis) $2\pi \int_c^d x \frac{ds}{dy}\, dy$ (about the y-axis)	$2\pi \int_{t_1}^{t_2} y \frac{ds}{dt}\, dt$ (about the x-axis) $2\pi \int_{t_1}^{t_2} x \frac{ds}{dt}\, dt$ (about the y-axis)	$2\pi \int_{\theta_1}^{\theta_2} r \sin\theta \frac{ds}{d\theta}\, d\theta$
Volume of revolution (V)	$\pi \int_a^b y^2\, dx$ (about the x-axis) $\pi \int_c^d x^2\, dy$ (about the y-axis)	$\pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt}\, dt$ (about the x-axis) $\pi \int_{t_1}^{t_2} x^2 \frac{dy}{dt}\, dt$ (about the y-axis)	$\frac{2\pi}{3} \int r^3 \sin\theta\, d\theta$ (about the line $\theta = 0$ or x-axis) $\frac{2\pi}{3} \int r^3 \cos\theta\, d\theta$ (about the line $\theta = \pi/2$ or y-axis)

UNIT - VI DIFFERENTIAL EQUATIONS (D.E)

Methods of solving the D.E at a glance

General form of the D.E : $M(x, y)\,dx + N(x, y)\,dy = 0$

	Form of the D.E	Method of solving / solution
I	Variables separable form (Recapitulation)	
1.	$f(x)\,g(y)\,dx + F(x)\,G(y)\,dy = 0$	Divide by $g(y)\,F(x)$ and integrate
2.	$\frac{dy}{dx} = f(ax + by + c)$	Put $ax + by + c = t$
3.	$\frac{dy}{dx} = \frac{(ax + by) + c}{k(ax + by) + c'}$	Put $ax + by = t$
II	**Homogeneous form**	
1.	$M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree with or without the involvement of terms with (y/x)	Write the D.E in the form $\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$ and put $y = vx$

	If homogeneous functions are involved with x/y	Write $\frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)}$ and put $x = vy$
2.	$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$, $\frac{a}{a'} \neq \frac{b}{b'}$	Put $x = X+h, y = Y+k$ With proper choice of h and k the D.E reduces to a homogeneous D.E in X and Y. Put $Y = VX$ and solve.
III	**Exact form**	
1.	$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ must be satisfied.	$\int M\,dx + \int N(y)\,dy = c$ is the solution.
2.	When $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then	Multiply the D.E with I.F to make it exact.
	(a) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$	$e^{\int f(x)\,dx}$ is the I.F
	$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$	$e^{-\int g(y)\,dy}$ is the I.F
	(b) $y f(xy)\,dx + x g(xy)\,dy = 0$	$\frac{1}{Mx - Ny}$ is the I.F
	(c) M and N involving terms of the form $x^a y^b$	$x^a y^b$ is the I.F where a and b are found such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
3.	Identifying the standard exact differentials and putting the D.E in the form $c_1 d\,[f_1(x, y)] + c_2 d\,[f_2(x, y)] + \cdots = 0$	$c_1 f_1(x, y) + c_2 f_2(x, y) + \cdots = c$ is the solution on integration
IV	**Linear form**	
1.	$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.	**Solution :** $y e^{\int P\,dx} = \int Q e^{\int P\,dx} dx + c$
2.	$\frac{dx}{dy} + Px = Q$ where P and Q are functions of y.	**Solution :** $x e^{\int P\,dy} = \int Q e^{\int P\,dy} dy + c$
3.	$f'(y)\frac{dy}{dx} + f(y)P = Q$ where $P = P(x)$ and $Q = Q(x)$	Put $f(y) = t$ and differentiate $w.r.t$ x.
4.	$f'(x)\frac{dx}{dy} + f(x)P = Q$ where $P = P(y)$ and $Q = Q(y)$	Put $f(x) = t$ and differentiate $w.r.t$ y.

5.	$\frac{dy}{dx} + Py = Q\,y^n$ where $P = P(x)$ and $Q = Q(x)$	Divide by y^n and put $y^{1-n} = t$ and diff. w.r.t x.
6.	$\frac{dx}{dy} + Px = Q\,x^n$ where $P = P(y)$ and $Q = Q(y)$	Divide by x^n and put $x^{1-n} = t$ and differentiate $w.r.t\ y$.

UNIT - VII LINEAR ALGEBRA - 1

Inverse of a square matrix A

$$A^{-1} = \frac{1}{|A|}(Adj\,A)$$

Normal form / canonical form of a matrix

(i) I_r (ii) $[I_r, 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

where I_r is the identity matrix of order r.

Rank of a matrix $A : \rho(A)$

- $\rho(A)$ = The number of nonzero rows in the row echelon/normal form of A

Given a matrix A there always exist non singular matrices P and Q such that PAQ is in the normal form.

Consistency of a system of equations $AX = B$

- The system is consistent if $\rho[A] = \rho[A:B]$
- The system will have *unique solution* if $\rho[A] = \rho[A:B] = n$, n being the number of unknowns.
- The system will have *infinite / many solutions* if $\rho[A] = \rho[A:B] = r < n$
- The system is *inconsistent* (does not have solution) if $\rho[A] \neq \rho[A:B]$
- The system $AX = 0$ will have trivial solution $(x_1 = 0 = x_2 = \cdots x_n)$ if $\rho[A] = \rho[A:B] = n$. The system will have nontrivial, infinite number of solutions if $\rho[A] = \rho[A:B] = r < n$
- *Gauss elimination method* : In $[A:B]$, A is reduced to the upper triangular form.
- *Gauss Jordan method* : In $[A:B]$, A is reduced to the diagonal form.

UNIT - VIII LINEAR ALGEBRA - 2

Eigen values and Eigen vectors of a square matrix A

- $|A - \lambda I| = 0$ will give the charecteristic equations of A. Eigen vlues are obtained on solving this equation.
- $[A - \lambda I][X] = [0]$ represents a system of equations. On solving this system of equations, eigen vector corresponding to each of the eigen value λ is obtained.

Similarity of matrices and Diagonalisation

- If $B = P^{-1} AP$ then B is said to be similar to A, where A and B are square matrices and P is a nonsingular matrix.
- $P^{-1} AP = D = Diag(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A (A *being a square matrix of order 3*)
- $A^n = P D^n P^{-1}$ where $D^n = Diag(\lambda_1^n, \lambda_2^n, \lambda_3^n)$

Quadratic Form (Q.F)

- Quadratic form can be reduced to canonical form (*sum of squares*) by congruent / orthogonal transformation.

  * Rank (r) of the Q.F = Rank of the matrix of the Q.F
  * Index (p) of the Q.F = Number of positive terms in the canonical form
  * Signature of the Q.F = The difference between the number of positive and negative terms in the canonical form

Nature of the Quadratic Form

	Condition	Nature of Q.F	Canonical form	Remark on canonical form
1.	$r = n, p = n$	Positive definite	$y_1^2 + y_2^2 + \cdots + y_n^2$	Only positive terms (n terms).
2.	$r = n, p = 0$	Negative definite	$-y_1^2 - y_2^2 \cdots - y_n^2$	Only negative terms (n terms)
3.	$r = p, p < n$	Positive semi definite	$y_1^2 + y_2^2 + \cdots + y_r^2$	Only positive terms (r terms)
4.	$r < n, p = 0$	Negative semi-definte	$-y_1^2 - y_2^2 - \cdots - y_r^2$	Only negative rerms (r terms)

In all other cases the Q.F is said to be indefinite. Indefinite Q.F contains both positive and negative terms.

ENGINEERING MATHEMATICS-I

Dr.K.S.C.

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- Lecture type of explanation
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Dr.K.S.Chandrashekar, popularly recognized as **Dr. K.S.C.** started his teaching career at the age of 20, soon after his post graduation from the Bangalore University securing second rank. Along with his involvement in teaching undergraduate and post graduate courses he worked for the doctorate degree also and secured it from the Karnatak University, Dharwad. Dr.K.S.C. has a continuous teaching experience of 37 years at the National Institute of Engineering, Mysore. In addition to being a highly popular teacher, he is the author of several popular student friendly textbooks on Engineering Mathematics for Mysore, Kuvempu and Visvesvaraya Technological Universities. He is the recipient of Rotary Datta Teachers Award during the year 2003. He has received felicitations from various organizations in Mysore.

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